Riemann Surfaces and Complex analytic geometry

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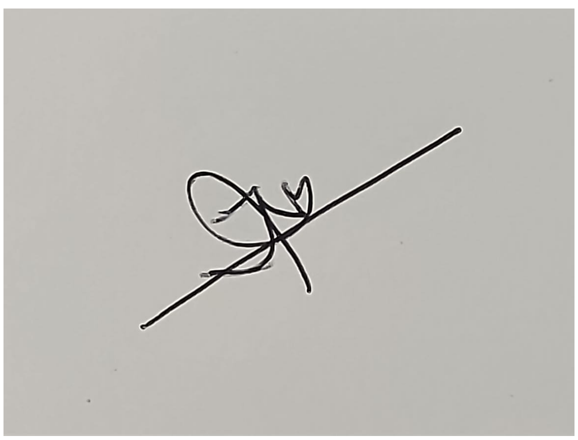

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This is to certify that the thesis titled submitted by **Navprabhat Semalti** Roll No.**20MS036** dated , a student of the **Department of Mathematics and Statistics** of the BS-MS program of IISER Kolkata, is based upon his own research work under my supervision. I also certify, to the best of my knowledge, that neither the thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before. In my opinion, the thesis fulfills the requirement for the award of the degree of BS-MS.

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ABSTRACT

Name of the student: **Navprabhat Semalti** Roll No: **20MS036**
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The goal of this thesis is to study some key aspects of the theory of 1-complex manifolds called "Riemann surfaces" that are also 2-smooth manifolds. Riemann surfaces are nicer to study in the sense that behaviour of holomorphic functions on the complex plane can be translated onto them upto some degree, this makes their theory slightly less different and less challenging than higher dimensional complex manifolds. In general, people try to study Riemann surfaces as compact Riemann surfaces and non-compact Riemann surfaces(Open Riemann surfaces) since the phenomenon obtained as such vary depending upon this choice. We have studied the behaviour of holomorphic mappings of Riemann surfaces, which when restricted to compact connected settings give a quantification by Riemann-Hurwitz formula[4]. Along the way, for broader techniques on Riemann surfaces we have worked with holomorphic line bundles that allows us to use L^2 theory on the space of their sections[4]. These techniques further allows one to probe into the phenomenon of analytic continuation which is also studied through del-bar problem on domains in $\mathbb{C}^n$[4].

Contents

Chapter 1

Riemann Surfaces

1.1 Introduction

The theory of functions of several complex variables is quite varied and interesting and various nice properties and phenomenon can be directly generalized from the theroy of one complex variable, however for studying at more general levels and better disguised phenomenon theory of complex variables has been generalized to somewhat special domains called "complex manifolds". Here we deal with a complex manifold of complex dimension n=1 called a Riemann surface where theory of one complex variables has much nicer generalizations.

Holomorphic Function(Definition)- Let $\Omega \subset \mathbb{C}^n$ be an open set. We say a complex valued function $f : \Omega \to \mathbb{C}$ is holomorphic in several variables if $\dfrac{\partial f}{\partial \overline{z_i}} = 0$ on Ω for $i = 1, 2, ..n$.

Holomorphic Mapping(Definition)- Let $\Omega \subset \mathbb{C}^n$ be a open set. We say $F : \Omega \to$

$\mathbb{C}^n$ is holomorphic mapping if each component functions $F_i : \Omega \to \mathbb{C}$ are holomorphic i.e. $\dfrac{\partial F_i}{\partial \overline{z_j}} = 0$ for $i, j \in \{1, 2, ..n\}$.

Biholomorphic Mapping(Definition)- Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be two open sets. We say a holomorphic mapping $F : \Omega_1 \to \Omega_2$ is a biholomorphism if F is bijective and $F^{-1} : \Omega_2 \to \Omega_1$ is also holomorphic in several complex variables.

Let X be a Hausdorff topological space.

Complex Chart(Definition)- Let $U \subset X$ be a open set and $\phi : U \to \mathbb{C}^n$ be a homeomorphism. We say the pair (U, ϕ) is a complex chart.

Compatibility of coordinate charts(Definition)- Let (U, ϕ) and (V, ψ) be two complex charts such that $U \cap V \neq \phi$. We say that the complex charts are compatible if $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a biholomorphic mapping.

Complex Atlas(Definition)-
We say a collection of complex charts $\mathcal{A}_X = \{(\phi_\alpha, U_\alpha) | \alpha \in I\}$ is a complex atlas if:
i) $\bigcup_{\alpha \in I} U_\alpha = X$
ii)(ϕ_α, U_α) and (ϕ_β, U_β) are compatible for any $\alpha, \beta \in I$

Complex Manifold(Definition)- Let X be a topological Hausdorff space, if X admits a complex atlas $\mathcal{A}_X$ we say X is a complex manifold of complex dimension n.

Riemann surface(Definition)- A complex manifold of complex dimension 1 is called a Riemann surface.

1.2 Construction of some Compact Riemann surfaces

1. The Riemann Sphere($\mathbb{P}^1$)-

Consider the set $X = \{(u,v,w) \in \mathbb{R}^3 | u^2+v^2+w^2 = 1\} \subset \mathbb{R}^3$[4]. We know that X is a topological Hausdorff space in the subspace topology in $\mathbb{R}^3$. Furthermore, it is compact as well.

Denote the north pole of the sphere as $e_N = \{(0,0,1)\}$ and south pole as $e_S = \{(0,0,-1)\}$. Take the stereographic projection about e_N which sends a point on a sphere say $(u,v,w) \in X$ to a point on the plane say $(x,y,0)$

By the equation of straight line passing through e_N and joining (u,v,w) and $(x,y,0)$ we get that

$$\frac{x-0}{u-0} = \frac{y-0}{v-0} = \frac{0-1}{w-1}$$

and $(u,v,w) \in X$ goes to $(\frac{u}{1-w}, \frac{v}{1-w}) \in \mathbb{R}^2$

Similarly taking stereographic projection by e_S,

$$\frac{x-0}{u-0} = \frac{y-0}{v-0} = \frac{0-(-1)}{w-(-1)}$$

we get that $(u,v,w) \in X$ goes to $(\frac{u}{w+1}, \frac{v}{w+1}) \in \mathbb{R}^2$

Let $U_N = X - \{(0,0,1)\}$ and $U_S = X - \{(0,0,-1)\}$ be two open sets in X. We define $\phi_N : U_N \to \mathbb{C}$ and $\phi_S : U_S \to \mathbb{C}$ as follows:

$$\phi_N(u,v,w) = \frac{u+iv}{1-w}$$

$$\phi_S(u,v,w) = \frac{u - iv}{1 + w}$$

Since $w \neq 1$ for $(u,v,w) \in U_N$ and $w \neq -1$ for $(u,v,w) \in U_S$, ϕ_N and ϕ_S are well-defined.

Let $z = x + iy = \phi_N(u,v,w)$, we want to find $(u,v,w) = (u(x,y), v(x,y), w(x,y))$ We have that $u = x(1-w)$ and $v = y(1-w)$, furthermore since $(u,v,w) \in X$

$$u^2 + v^2 + w^2 = 1$$

$$w^2 = 1 - (x^2 + y^2)(1-w)^2$$

$$w^2 = (1 - x^2 - y^2)(1-w)^2$$

$$(x^2 + y^2 + 1)w^2 - 2(x^2 + y^2)w + (x^2 + y^2 - 1) = 0$$

We get that,

$$w = \frac{2(x^2 + y^2) \pm \sqrt{4(x^2 + y^2)^2 - 4(x^2 + y^2 + 1)(x^2 + y^2 - 1)}}{2(x^2 + y^2 + 1)}$$

$$w = \frac{x^2 + y^2 \pm 1}{x^2 + y^2 + 1}$$

On U_N we will have $w = \dfrac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$, $u = \dfrac{2x}{x^2 + y^2 + 1}$ and $v = \dfrac{2y}{x^2 + y^2 + 1}$

We define $\phi_N^{-1}(x + iy) = (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1})$ and since ϕ_N^{-1} is well-defined, ϕ_N is bijective. By componentwise continuity, we can see that both ϕ_N and ϕ_N^{-1} are continuous. Hence $\phi_N : U_N \to \mathbb{C}$ is a homeomorpism.

Consider the transition map from coordinate chart (U_N, ϕ_N) to coordinate chart

(U_S, ϕ_S), $\phi_S \circ \phi_N^{-1} : \phi_N(U_N \cap U_S) \to \phi_S(U_N \cap U_S)$ is given by

$$z = x + iy \to \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right) \to \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{z} = \frac{1}{z}$$

Since $\phi_N(U_N \cap U_S) = \phi_S(U_N \cap U_S) = \mathbb{C} - \{0\}$ the above transition function is holomorphic with holomorphic inverse.

Note that $U_N \cup U_S = X$, so the complex atlas $\{(\phi_N, U_N), (\phi_S, U_S)\}$ gives a biholomorphic structure on X. Hence X is a Riemann surface.

2. The Complex torus

Let ω_1 and $\omega_2 \in \mathbb{C}$ be two $\mathbb{R}$-independent vectors, we look at the free module generated by them $L = \{a\omega_1 + b\omega_2 | a, b \in \mathbb{Z}\}$[4]. Note that $L \subset \mathbb{C}$ is a commutative subgroup in $\mathbb{C}$.

We define X to be the quotient group of L in $\mathbb{C}$ i.e.$X = \mathbb{C}/L$. So there exists a natural projection map $\pi : \mathbb{C} \to X$ given by $z \to z + L$ which is a group homomorphism. Note that π is surjective.

We say $U \subset X$ is open if $\pi^{-1}(U)$ is open in $\mathbb{C}$. This gives quotient topology on X via the quotient map π. Thus, $\pi : \mathbb{C} \to X$ is a continuous map. Furthermore, we claim that π is an open map.

Let $\widetilde{U} \subset \mathbb{C}$ be an open set. Then, $\pi^{-1}(\pi(\widetilde{U})) = \bigsqcup_{\omega \in L} (\widetilde{U} + \omega)$

Since $\widetilde{U} + \omega$ is open in $\mathbb{C}$, for all $\omega \in L$, we have that $\pi(\widetilde{U})$ is open.

Now, choose $\epsilon < \frac{1}{2}.min\{|\omega_1|, |\omega_2|\}$ then $\mathbb{D}(0, 2\epsilon) \cap L = \{0\}$.

Let $z \in \mathbb{C}$ we claim that $\mathbb{D}(z, \epsilon) \cap \mathbb{D}(z, \epsilon) + \omega = \phi$, for all $\omega \in L - \{0\}$. If $z' \in \mathbb{D}(z, \epsilon) \cap \mathbb{D}(z, \epsilon) + \omega$, then $|z'| < \epsilon$ and $|z' + \omega| < \epsilon$ implying $|w| = |w + z' - z'| \leq$

$|w+z'|-|z'|<2\epsilon$. Hence, $w \in \mathbb{D}(0,2\epsilon) \cap L$ so $w=0$.

We define $\pi_z := \pi|_{\mathbb{D}(z,\epsilon)} : \mathbb{D}(z,\epsilon) \to \pi(\mathbb{D}(z,\epsilon))$ Since $\pi_z^{-1}(\pi(\mathbb{D}(z,\epsilon))) = \mathbb{D}(z,\epsilon)$, it is bijective hence a homeomorphism.

The collection $\mathcal{A} = \{(\pi(\mathbb{D}(z,\epsilon)), \pi_z^{-1}) | z \in \mathbb{C}\}$ is an atlas for X.

Theorem- Let X be a Riemann surface, then X is an orientable 2-manifold. Proof- Let (U,ϕ) and (ψ,V) be two coordinate charts. Then, $\psi \circ \phi^{-1} : \phi(U) \to \psi(V)$ is a biholomorphism.[4]

$Det(D(\psi \circ \phi^{-1})) = |(\psi \circ \phi^{-1})'|^2 > 0$ on $U \cap V$.

Hence, X is an orietable 2-manifold.

Chapter 2

On holomorphic mappings of Riemann Surfaces

Holomorphic Mapping (Definition)-

Let X and Y be Riemann surfaces. We say $F : X \to Y$ is holomorphic at $p \in X$ iff $\exists$ coordinate charts (U, ϕ) at $p \in X$ and (ψ, V) at $F(p)$ s.t. $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is holomorphic on $\phi(U)$.

We say $F : X \to Y$ is holomorphic on X if F is holomorphic at every point of X.

2.1 Properties of holomorphic mappings

Open Mapping Theorem- Let X, Y be connected Riemann surfaces and $F : X \to Y$ be a non-constant holomorphic mapping. Then, F maps open sets to open sets.[4]

Proof-Let $U \subset X$ be open we want to show that $F(U)$ is open. Consider $y \in F(U)$, then $\exists x \in U$ such that $F(x) = y$. Let (ϕ, U') and (ψ, V) be coordinate charts at $x \in U'$ and $y \in V$, then the map $\psi \circ F \circ \phi^{-1} : \phi(U') \to \psi(V)$ is holomorphic on $\phi(U)$.

Claim: $\widetilde{F} = \psi \circ F \circ \phi^{-1}$ is non-constant on $\phi(U)$

Suppose it's not true, then F is a constant map on U. This implies that U is both open and closed. By connectedness of X, $U = X$. This implies that F is constant on X, hence a contradiction. Let V be a connected neigbourhood of $\phi(U)$ containing x, then by open mapping theorem on V, $\widetilde{F} : V \to F(V)$ is an open mapping. Since, $F(V) \cap F(U)$ is an open neighbourhood of y in $F(U)$. This implies that y is an interior point, since y is arbitrary $F(U)$ is open.

Theorem- Let X and Y be Riemann surfaces, where X is connected and $F : X \to Y$ is a non-constant holomorphic mapping. Then, for each $y \in Y$, $F^{-1}(y)$ is a discrete set in X.[4]

Proof-

Let $y \in Y$,assume $F^{-1}\{y\} \neq \phi$, otherwise it is trivial. Suppose it is not true i.e. there exists $x \in F^{-1}(y)$ such that x is a limit point of the set $F^{-1}\{y\}$.

Since F is holomorphic, $\exists$ (ϕ, U_x) and (ψ, V_y) such that $\psi \circ F \circ \phi^{-1} : \phi(U_x) \to \psi(V_y)$ is a holomorphic function. By translation we can choose (ψ, V_y) such that $\psi(y) = 0$. Since x is a limit point of $F^{-1}\{y\}$ $\exists$ a sequence $\{x_n\}$ in U_x such that $x_n \to x$. This implies $\phi(x_n)$ converges to $\phi(x)$, where $\psi \circ F \circ \phi^{-1}(\phi(x_n)) = 0$ and $\psi \circ F \circ \phi^{-1}(\phi(x)) = 0$. Hence, $\psi \circ F \circ \phi^{-1} \equiv 0$ on $\phi(U)$.

This implies U is an open and closed set in X, since X is connected, $U = X$ and F is a constant function this contradicts the fact that F is non-constant.

Normal Form Theorem-

Let X,Y be Riemann surfaces and $F : X \to Y$ be a holomorphic mapping between them. Let $x \in X$, then $\exists$ coordinate charts (U, ϕ) at $x \in X$ & (V, ψ) at $F(x)$ and a positive integer m_x such that :

i) $\phi(x) = 0$ and $\psi(F(x)) = 0$

ii) $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is given by $\psi \circ F \circ \phi^{-1}(z) = z^{m_x}$

iii) The integer m_x is unique for each $x \in X$ [4]

Proof-

Let $x \in X$ and $y = F(x) \in Y$, then there exists coordinate charts (U, ϕ) at x and (V, ψ) at y such that $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is holomorphic. By setting $\phi'(x') = \phi(x') - \phi'(x)$ and $\psi'(y') = \psi(y') - \psi(y)$ we can choose coordinate charts such that $\widetilde{F} = \psi' \circ F \circ \phi^{-1} : \phi'(U) \to \psi'(V')$ has a zero at $z = 0 \in \phi'(U)$

This implies that $\exists$ a positive integer m such that $\widetilde{F}(z) = z^m G(z)$ for all $z \in \phi'(U)$ where $G : \phi(U) \to \mathbb{C}$ is a holomorphic function such that $G(0) \neq 0$.

By continuity of G, there exists $r > 0$ such that $G(z) \neq 0$ on $D(0, r) \subset \phi'(U)$. Hence, $G(z) = e^{H(z)}$, for some holomorphic function H on $D(0, r)$. It follows that $\widetilde{F} : D(0, r) \to \psi(V)$ can be written as $z \to z^m e^{H(z)}$.

Now, define $\widetilde{\phi}(z) = zexp\left(\dfrac{H(z)}{m}\right)$ for $z \in D(0, r)$

Note that $\widetilde{\phi}(z) = exp(\dfrac{H(z)}{m}) + ze(\dfrac{H(z)}{m})\dfrac{H'(z)}{m}$ and $\widetilde{\phi}'(0) = 1$.

Since $\widetilde{\phi}'$ is non-zero at $z = 0$, by inverse function theorem $\exists\, U' \subset D(0, r)$ such that $\widetilde{\phi}$ is invertible on U'.

$\psi' \circ F \circ (\phi'^{-1} \circ \widetilde{\phi}^{-1})(\tau) = \widetilde{F}(\widetilde{\phi}^{-1}(\tau)) = \tau^m$, for all $\tau \in \widetilde{\phi}(U')$

ii) Let (ϕ_1, U_1) ,(ψ_1, V_1) and (ϕ_2, U_2),(ψ_2, V_2) be two coordinates satisfying Normal

Form Theorem,i.e. $\widetilde{F_1} = \psi_1 \circ F \circ \phi_1 : z \to z^{m_1}$ and $\widetilde{F_2} : \psi_2 \circ F \circ \phi_2^{-1} : z \to z^{m_2}$.

Note that $\widetilde{F_1} = \psi_1 \circ \psi_2^{-1} \circ \widetilde{F_2} \circ \phi_2 \circ \phi_1^{-1}$. Denote $h_1 = \phi_2 \circ \phi_1^{-1}$ and $h_2 = \psi_1 \circ \psi_2^{-1}$ $z^{m_1} = h_2((h_1(z))^{m_2})$, By power series expansion we can see that $z^{m_1} = (\sum\limits_q \frac{a_q}{q!}(\sum\limits_p \frac{b_p z^p}{p!})^{m_2 q}) = z^{m_2}(\sum\limits_q \frac{a_q}{q!} \sum\limits_p \frac{b_p z^{p-1}}{p!})^{m_2 q}$.This implies that $m_1 \leq m_2$.

Since, h_1 and h_2 are biholomorphic, their inverse exists say g_1 and g_2 for $\widetilde{F_2} = g_2 \circ \widetilde{F_1} \circ g_1$ we will get $m_2 \leq m_1$.Hence, $m_1 = m_2$.

Multiplicity (Definition)- Let $F : X \to Y$ be a holomorphic mapping of Riemann surfaces. Let $x \in X$, the integer m_x is called the multiplicity of F at $x \in X$ denoted by $Mult_x(F)$.
Note that since F is holomorphic, $Mult_x(F) \geq 0$.

Ramification Point(Definition)- Let $F : X \to Y$ be a holomorphic mapping of Riemann surfaces. We say $x \in X$ is a ramification point iff $Mult_x(F) \geq 2$.

Branch Point(Definition)- Let $F : X \to Y$ be a holomorphic mapping of Riemann surfaces. We say $y \in Y$ is a branch point if for some $x \in F^{-1}\{y\}$,$Mult_x(F) \geq 2$.

Theorem- Let X and Y be connected Riemann surfaces and $F : X \to Y$ be a holomorphic mapping.Then,
i)The set of ramification points $R(F) \subset X$ is discrete.
ii)The set of branch points $B(F) \subset Y$ is discrete.

Proof- i)Let x be a limit point of $R(F)$ and (U_x, ϕ), $(V_{F(x)}, \psi)$ be coordinate charts s.t. $\psi \circ F \circ \phi^{-1}(z) = z^m$ on $\phi(U_x)$ with $m_x \geq 2$. Then, $\exists x' \in U_x$ such that $Mult_{x'}(F) \geq 2$. Denote $y' = F(x')$

Since X is Hausdorff, for each $\widetilde{x} \in F^{-1}(y')$ we can choose neighbourhood $U_{\widetilde{x}}$ such that $F^{-1}(y') \cap U_{\widetilde{x}} = \{\widetilde{x}\}$. By Normal Form Theorem at $\widetilde{x}$ and y' for some $y'' \in V_{y'}$, $F^{-1}(y'') \cap U_{\widetilde{x}}$ has $m_{\widetilde{x}}$ many preimages in $U_{\widetilde{x}}$.

$$|F^{-1}(y'')| \cap U_{x'} = \sum_{x \in F^{-1}(y') \cap U_x} |F^{-1}(y'')| \cap U_{\widetilde{x}} \geq 2 + Mult_x(F) - 1$$

This is a contradiction since $|F^{-1}(y'')| \cap U_{x'} = Mult_{x'}(F)$

ii)The branch points are precisely the image of ramification points, since F is continuous $R(F)$ is discrete.

2.2 Holomorphic Mappings of Compact and Connected Riemann surfaces

Theorem- Let X and Y be connected Riemann surfaces and $F : X \to Y$ be a non-constant holomorphic mapping. If X is compact, then Y is compact and F is surjective.[4]

Proof- Since F is a non-constant holomorphic mapping of connected Riemann surfaces it is an open map. This implies that $F(X)$ is open in Y. Furthermore, since X is compact $F(X)$ is compact from continuity of F. Since Y is a connected set $F(X)$ is non-empty $F(X) = Y$. Hence, F is surjective and Y is compact.

Theorem- Let X and Y be compact, connected Riemann surfaces and $F : X \to Y$ is a holomorphic mapping on X. Then, $Deg(F) : Y \to \mathbb{Z}$ defined by $Deg(F)(y) = \sum_{x \in F^{-1}(y)} Mult_x(F)$ is constant.[4]

Proof-

Let $Y_n = \{y \in Y | Deg(F)(y) \geq n)\} = Deg(F)^{-1}[n, \infty)$,for some $n \in \mathbb{N}$.

We want to show that Y_n is both open and closed.

i) Let $y \in Y_n$ and $x \in F^{-1}(y)$,Then by Normal form theorem $\exists$ coordinate charts (U, ϕ) at x and (V, ψ) at y s.t. $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is given by $z \to z^m$ and $\psi(y) = 0 = \phi(x)$.

Let $y' \in V$, then since $\phi : U_x \to \phi(U_x)$ is bijective

$|F^{-1}(y') \cap U_x| = |(\psi \circ F \circ \phi^{-1})^{-1}(\psi(y'))| = m_x \geq |(\psi \circ F \circ \phi^{-1})^{-1}(0)| = |F^{-1}(y) \cap U_x|$.

Then, by definition of degree of a holomorphic map

$Deg(F)(y') = \sum_{x \in F^{-1}(y')} Mult_x(F) \geq \sum_{x \in F^{-1}(y')} 1 \geq \sum_{i=1}^{k} |F^{-1}(y') \cap U_{x_i}| = \sum_{i=1}^{k} m_{x_i}$

$\geq \sum_{i=1}^{k} |F^{-1}(y) \cap U_{x_i}| = Deg(F)(y) \geq n.$
Since $Deg(F(y)) \geq n$, we have that $Deg(F(y')) \geq n$,for all $y' \in V$.

ii)Let $\{y_k\}_{k\geq 1}$ be a sequence in Y_n such that $y_k \to y$ in Y. We want to show that $y \in Y_n$ i.e. $Deg(F)(y) = \sum_{x \in F^{-1}(y)} Mult_x(F) \geq n$.

Remove the branch points from the sequence y_k so that $Mult_x(F) = 1$, for all $x \in F^{-1}(y_k)$. Since $Deg(F)(y_k) \geq n$,it implies that $F^{-1}(y_k) \supset \{x_{1,k}, x_{2,k}, ...x_{n,k}\}$.
Since $\{x_{i,k}\}_{k\geq 0}$ are sequences in X, $\exists$ convergent subsequences say $\{x_{i,k_j}\}_{j\geq 0}$ converging to $x_i \in X$,for all $i = 1, 2, ...n$.
Furthermore, $F(x_i) = \lim_{j\to\infty} F(x_{i,k_j}) = \lim_{k\to\infty} y_k = y$, for all $i = 1, 2..n$

Case I- Suppose all $x_i's$ are different. Then,
$Deg(F)(y) = \sum_{x \in F^{-1}(y)} Mult_x(F) \geq \sum_{i=1}^{n} Mult_{x_i}(F) \geq n$

Case II- Let j-many of $x_i's$ are same say $x_{i_1} = x_{i_2} = ... = x_{i_j} = x$
$Deg(F)(y) = \sum_{x \in F^{-1}(y)} Mult_x(F) \geq \sum_{i=1}^{n-j} Mult_{x_i}(F) + Mult_x(F)$

By Normal Form Theorem at $x \in X$, $\exists$ coordinate charts (U_x, ϕ) and (V_y, ψ) with $V_y \subset F(U_x)$ such that $\psi \circ F \circ \phi^{-1} : \phi(U_x) \to \psi(V_y)$ is given by $z \to z^m$
Since x is a limit point in X, $\exists N \in \mathbb{N}$ s.t. $x_{i,k_j} \in U_x$,for all $j \geq N$,$i = 1, 2, .., j$. This implies $j \leq |F^{-1}(y_k) \cap U_x| = m$. Hence,
$Deg(F)(y) = \sum_{x \in F^{-1}(y)} Mult_x(F) \geq \sum_{i=1}^{n-j} Mult_{x_i}(F) + Mult_x(F) \geq n.$

By definition, $Y_n \subseteq Y_{n-1}$. Since Y is connected, Either $Y_n = \phi$ or $Y_n = Y$. Let $y \in Y$ with $Deg(F)(y) = k$, then $y \in Y_k$ but $y \notin Y_{k+1}$.This implies that $Y_{k+1} = \phi$ and $Y_n = Y$ for all $n \leq k$.

This means $Deg(F)(y) \geq k$, for all $y \in Y$ and $Deg(F)(y) < k$ for all $y \in Y$. Hence, $Deg(F)(y) = k$,for all $y \in Y$.

Theorem(Riemann-Hurwitz Formula)-

Let X and Y be compact and connected Riemann surfaces & $F : X \to Y$ is a holomorphic mapping. Then,

$$2g(X) - 2 = (2g(Y) - 2)Deg(F) + \sum_{x \in X}(Mult_x(F) - 1)$$

[4]

Proof- Let $\mathcal{F}_Y = \{T_i | i = 1, 2, .., n\}$ be a triangulation for Y[1]. Let $y \in B(F)$be a branch point, then $y \in T_{i_0}$, for some $i_0 = 1, ..n$; If $y \in Int(T_{i0})$ join y with each vertex of T_{i0}. This decomposes T_{i0} into three triangles. If $y \in Edge(T_{i0})$, then $y \in T_{i0} \cap T_{i1}$ joint y with vertices of T_{i0} and T_{i1}.

Hence,we can assume a tringulation exists such that all ramification points are vertices in the triangulation.

If there exists $y_1, y_2 \in B(F)$ such that there is an edge $[y1, y2]$between them, then choose a point $y' \in [y_1, y_2]$ and join it with the vertices of the triangle it is contained in. Hence, this implies that there is no edge with both points as ramification points.

Suppose the tringulation of Y has V vertices, E edges and F faces.

Since $d = Deg(F)(y) = \sum_{x \in F^{-1}(y)} Mult_x(F)$. For $y \in V_u$,$|F^{-1}(y)| = d$ and for $y \in V_b$,$|F^{-1}(y)| = d - \sum_{x \in F^{-1}(y)} (Mult_x(F) - 1)$.

The no of vertices in the pullback triangulation is given by[4]:

$$W = \sum_{y \in V_u} |F^{-1}(y)| + \sum_{y \in V_b} |F^{-1}(y)|$$

$$W = \sum_{y \in V_u} d + \sum_{y \in V_b} (d - \sum_{x \in F^{-1}(y)} (Mult_x(F) - 1))$$

$$W = dV - \sum_{x \in F^{-1}(y)} (Mult_x(F) - 1)$$

By definition of Euler Characteristic,

$\chi(X) = V(X) - E(X) + F(X) = W - dE + dF = d(V - E + F) + \sum_{x \in F^{-1}(y)} (Mult_x(F) - 1)$

$\chi(X) = d.\chi(Y) + \sum_{x \in F^{-1}(y)} (Mult_x(F) - 1)$

BY theorem on classification of orientable surfaces[2],

$(2 - 2g(Y)) = d(2 - 2g(X)) + \sum_{x \in F^{-1}(y)} (Mult_x(F) - 1)$, where $g(X), g(Y)$ are genuses of the surfaces X, Y.

Ramification Degree-

Let $F : X \to Y$ be a non-constant map of compact and connected Riemann surfaces. We say that the integer $R(F) = \sum_{x \in X} (Mult_x(F) - 1)$ is the Ramification degree of the non-constant holomorphic map of Riemann surfaces.

Note that $R(F)$ is an even integer.

Corollary-

Let $F : X \to Y$ is a non-constant holomorphic map of compact connected Riemann surfaces. Then, $g(Y) \leq g(X)$.[4]

Proof- Since $Mult_x(F) \geq 0$, we get $Deg(F) \geq 0$, $R(F) \geq 0$. By Riemann-Hurwitz formula, $2g(X) - 2 \geq (2g(Y) - 2)Deg(F) \geq 2g(Y) - 2$, This implies $g(X) \geq g(Y)$.

Corollary(Sufficient condition for a unramified map)-

Let X, Y, F be as above. If $g(X) = g(Y)$, then F is unramified.[4]

Proof- Put $g(X) = g(Y)$ in the Riemann-Hurwitz formula we get that $R(F) = 0$.

Corollary(Necessary condition for a unramified map)-

Let $X, Y, F : X \to Y$ be as above. If F is unramified, then [4]

i)$g(X) = 1 \implies g(Y) = 1$

ii) If $g(X) > 1$ and $Deg(F) > 1$, then $g(Y) > 1$ and $Deg(F)|(g(X) - 1)$.

Proof-

By rearranging the Riemann-Hurwitz formula,

$$g(Y) = \frac{g(X)}{Deg(F)} + \frac{Deg(F) - 1}{Deg(F)} - \frac{R(F)}{2Deg(F)}$$

Put $R(F) = 0$,

$$g(Y) = \frac{g(X)}{Deg(F)} + \frac{Deg(F) - 1}{Deg(F)}$$

For $g(X) > 1$, $g(Y) > 1$ and for $g(X) = 1$, $g(Y) = 1$

Corollary(On holomorphic covering maps to Riemann sphere)-

Let X be a compact and connected Riemann surface and $F : X \to \mathbb{P}^1$ is a non-constant holomorphic map. If F is unramified, then F is necessarily an isomorphism.[4]

Proof-

Putting $R(F) = 0$ and $g(Y) = 0$ in the Riemann-Hurwitz formula, we get $Deg(F) = 1 - g(X)$.This implies that $Deg(F) \leq 1$ since $Deg(F) \geq 1$. $Deg(F) = 1$ and F is an isomorphism of Riemann surfaces.

Chapter 3

Complex Line Bundles on a Riemann Surface

3.1 Introduction

Complex Line Bundle(Definition)-

Let L and M be smooth manifolds. We say that L along with a map $\pi : L \to M$ is a complex line bundle if:

i) For every $p \in M$,there exists $U_x \subset M$ and $F_{U_x} : \pi^{-1}(U_x) \to U_x \times \mathbb{C}$ defined as $v \to (\pi(v), f_U(v))$ is a diffeomorphism.

ii) If U_α, U_β satisfies (i), s.t. $U_\alpha \cap U_\beta = \phi$, then $\exists g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ where, $F_{U_\alpha} \circ F_{U_\beta}^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}$ is given by $(x, \lambda) \to (x, g_{\alpha\beta}(\lambda))$

Note that the fiber $L_x = \pi^{-1}(x) = \{x\} \times \mathbb{C}$ is isomorphic to $\mathbb{C}$ as a complex vector space.

The 1-cocycle condition-

Consider $U_\alpha, U_\beta, U_\gamma \subset M$ satisfying (i) and $U_\alpha \cap U_\beta \cap U_\gamma \neq \phi$, then,

$(F_{U_\alpha} \circ F^{-1}_{U_\beta}) \circ (F_{U_\beta} \circ F^{-1}_{U_\gamma}) \circ (F_{U_\gamma} \circ F^{-1}_{U_\alpha}) = Id$ on $(U_\alpha \cap U_\beta \cap U_\gamma) \times \mathbb{C}$.

This implies that $g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = 1$ for each $x \in U_\alpha \cap U_\beta \cap U_\gamma$[4]

Morphism of Complex Line Bundle(Definition)-

Let (L, π) and (L', π') be two complex line bundles over M. We say a map $F : L \to L'$ is a morphism of complex line bundles if :

i) $\pi' \circ F \circ \pi = id_X$

ii)$F_x = F|_{L_x} : L_x \to L'_x$ is a map of complex vector spaces.

Isomorphism of Complex Line Bundles(Definition)-

We say that two complex line bundles (L, π) and (L', π') over the M are isomorphic if $\exists$ complex line bundle morphisms $F : L \to L'$ and $G : L' \to L$ such that $F \circ G = Id_{L'}$ and $G \circ F = Id_L$.

3.2 Constrution of a complex line bundle

Proppsition-(Primitive construction of complex line bundle)

Let $\{U_\alpha | \alpha \in I\}$ be an open cover of M and $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^* | \alpha, \beta \in I\}$ be a collection of transition data satisfying the 1-cocycle condition. Then,

i) $\exists$ a $\mathbb{C}$-line bundle $(L\{g_{\alpha\beta}\}, \pi)$ such that the transition data of L is $\{g_{\alpha\beta}\}$.

ii)If (L, π) is a complex line bundle with a trivializing cover $\{U_\alpha\}$ and transition data $\{g_{\alpha\beta}\}$, then $L \cong L\{g_{\alpha\beta}\}$ [4].

Proof-

Consider the space $Y = \bigsqcup_{\alpha \in I} U_\alpha \times \mathbb{C} \times \{\alpha\}$ equipped with disjoint union topology. We define a relation $\sim$ on Y by $(x, z, \alpha) \sim (x, w, \beta)$ iff $w = g_{\alpha\beta}(x)z$.

i)Reflexivity- $(x, z, \alpha) \sim (x, z, \alpha)$ since $z = 1.z = g_{\alpha\alpha}(x)z$

ii) Symmetry- If $(x, z, \alpha) \sim (x, w, \beta)$ then $z = g_{\alpha\beta}(x)w$ which implies $z = g_{\beta\alpha}(x)w$ since $g_{\alpha\beta}(x) \in \mathbb{C}^*$

iii)Transitivity- Let $(x, z, \alpha) \sim (x, w, \beta) \sim (x, \tau, \gamma)$, then $\tau = g_{\beta\gamma}(x)w = g_{\beta\gamma}(x)g_{\alpha\beta}(x)z$. Since $\{g_{\alpha\beta}\}$ satisfies the 1-cocycle condition $g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = 1$. This implies that $\tau = g_{\alpha\gamma}(x)z$ Hence, $(x, z, \alpha) \sim (x, \tau, \gamma)$.

Hence $\sim$ is an equivalence relation over Y. We define $L\{g_{\alpha\beta}\} = Y/\sim$

The quotient map $q : Y \to Y/\sim$ given by $(x, z, \alpha) \to [(x, z, \alpha)]$ induces the quotient topology on $L\{g_{\alpha\beta}\}$.i.e. $\widetilde{U} \subset Y/\sim$ is open iff $q^{-1}(\widetilde{U})$ is open in Y.

We define the map $\pi : L\{g_{\alpha\beta}\} \to M$ by $[(x, z, \alpha)] \to x$

Then $q^{-1}(\pi^{-1}(U_\alpha)) = \bigsqcup_{\beta \in J} (U_\alpha \cap U_\beta) \times \mathbb{C} \times \{\beta\}$ where $J = \{\beta \in I | U_\alpha \cap U_\beta \neq \phi\}$. Since each of these are open, $\pi^{-1}(U_\alpha)$ is open in $L\{g_{\alpha\beta}\}$. Furthermore, we have $\bigcup_{\alpha \in I} \pi^{-1}(U_\alpha) = L\{g_{\alpha\beta}\}$. Hence, $\{\pi^{-1}(U_\alpha) | \alpha \in I\}$ forms an open cover of $L\{g_{\alpha\beta}\}$.

We define $F_{U\alpha} : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C} \times \{\alpha\}$ by $[(x, z, \alpha)] \to (x, z, \alpha)$ by definition, F_{U_α} is bijective. For $V \subset U_\alpha \times \mathbb{C} \times \{\alpha\}$(open), $q^{-1}(F_{U_\alpha}^{-1}(V)) = \bigsqcup_{\beta \in J} V \cap (U_\beta \times \mathbb{C} \times \{\beta\})$. This shows that F_{U_α} is a continuous map.

For $\widetilde{U} \subset \pi^{-1}(U_\alpha)$(open) $q^{-1}(\widetilde{U}) = \bigsqcup_{\beta \in J} \pi(\widetilde{U} \cap U_\beta) \times \Omega \times \{\beta\}$ for some $\Omega \subset \mathbb{C}$(open). Now, $F_{U_\alpha}(\widetilde{U}) = \pi(\widetilde{U}) \times \Omega \times \{\alpha\}$. Hence, F_{U_α} is an open map.

Now, $F_{U_\beta} \circ F_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C} \times \{\alpha\} \to (U_\alpha \cap U_\beta) \times \mathbb{C} \times \{\beta\}$ is given by

$(x, z, \alpha) \to [(x, z, \alpha)] = [(x, g_{\alpha\beta}(x)z, \beta)] \to (x, g_{\beta\alpha}(x)z, \beta)$

Since $F_{U_\beta} \circ F_{U\alpha}^{-1}$ is linear in both variables it is a diffeomorphism. Hence $(\pi^{-1}(U_\alpha), F_{U_\alpha})$ gives a collection of smoothly compatible coordinate chart on $L\{g_{\alpha\beta}\}$ giving it a smooth manifold structure.

ii) Let (L, π) be another complex line bundle with trivializing cover U_α and transition data $\{g_{\alpha\beta}\}$. Define $F : L\{g_{\alpha\beta}\} \to L$ by $[(x, z, \alpha)] \to F_{U_\alpha}^{-1}(x, z)$ where F_{U_α} is the local trivialization of L on U_α.

Since $F_{U_\beta} \circ F_{U\alpha}^{-1}(x, z) = (x, g_{\alpha\beta}(x)z)$ we get that $F_{U_\alpha}^{-1}(x, z) = F_{U_\beta}^{-1}(x, g_{\alpha\beta}(x)z)$ $F([(x, z, \alpha)]) = F([(x, g_{\alpha\beta}(x)z, \beta)])$ hence, F is well defined.

$F|\pi^{-1}(x)$ sends $[(x, z, \alpha)] + c[(x, w, \alpha)] = [(x, z + cw, \alpha)] \to F_{U_\alpha}^{-1}(x, z + cw) = F_{U_\alpha}^{-1}(x, z) + cF_{U_\alpha}^{-1}(x, w)$ for $c \in \mathbb{C}$. Hence, F is $\mathbb{C}$-linear in each fiber.

Similarly, the map defined as $G : L \to L\{g_{\alpha\beta}\}$ as $v_x \to [(x, f_{U_\alpha}(v_x), \alpha)]$ is a morphism of complex line bundles where $F_{U_\alpha} : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ is the local trivialization. Since F and G are inverses of each other, $L \cong L\{g_{\alpha\beta}\}$

3.3 Sections of Complex Line Bundles

Section(Definition)- Let (L, π) be a complex line bundle over M. We say a map $s : M \to L$ is a section if $\pi \circ s = id_M$.

Nowhere vanishing Section(Definition)- Let $U \subset M$ be open. We say $\xi \in \Gamma(U, L|_U)$ is a framing section over $U \subset M$ if there exists trivializing neighbourhood U'_x such that $F_{U'_x}(\xi(x)) \neq (x, 0)$ for all $x \in X$.

Well-definedness of nowhere vanishing section-

Let U and V be trivializing neighbourhoods containing x and suppose that ξ is framing in coordinates of U then,

$F_V(\xi(x)) = F_V \circ F_U^{-1} \circ F_U(\xi(x))$

$F_V(\xi(x)) = F_V \circ F_U^{-1} \circ F_U((x, z))$ for some $z \neq 0$ in $\mathbb{C}$

$F_V(\xi(x)) = (x, g_{VU}(x)z)$

Since $g_{VU}(x) \in \mathbb{C}^*$ ξ is well-defined.

Proposition- Let (L, π) be a line bundle over M. Then, $L \cong M \times \mathbb{C}$ iff $\exists$ a nowhere vanishing section $\xi : M \to L$ over M.[4]

Proof- i) Let $\phi : L \to M \times \mathbb{C}$ be a complex line bundle isomorphism. Define $\xi : M \to L$ by $\xi(x) = \phi^{-1}(x, 1)$. Since $\phi : L \to M \times \mathbb{C}$ is a diffeomorphism ξ is a nowhere vanishing section.

ii) Let $\xi \in \Gamma(M, L)$ be a nowhere vanishing section and $v \in L_x$ for some $x \in M$ $\exists U_x \subset M$ trivializing $F_{U_x}(\xi(x)) = (x, f_U(\xi(x)))$ since $\xi(x) \neq 0$, we can write $F_{U_x}(v) = (x, \lambda(v).f_U(\xi(x)))$ where $\lambda(v) = \dfrac{f_U(v)}{f_U(\xi(x))} \in \mathbb{C}$

If $V_x \subset M$ is another trivializing nbhd then $f_V(v') = g_{VU}(x) f_U(v')$ for all $v' \in L_x$, $\lambda'(v) = \dfrac{f_V(v)}{f_V(\xi(x))} = \dfrac{f_U(v)}{f_U(\xi(x))} = \lambda(v)$, hence λ is independent of choice of trivializing coordinates.

Define $F : L \to M \times \mathbb{C}$ by $v \to (\pi(v), \lambda(v))$

$F|_{L_x}(v_x + w_x) = (x, \lambda(v_x + w_x)) = (x, \lambda(v_x) + \lambda(w_x)) = F|_{L_x}(v_x) + F|_{L_x}(w_x)$

The map $G : M \times \mathbb{C} \to L$ given by $(x, z) \to v_x$ where $v_x = z.\xi_{U_x}(x)$ is inverse of F, hence $L \cong M \times \mathbb{C}$.

Action of $\mathbb{C}^*$-

Define $\mathbb{C}^* \times L \to L$ by $\lambda.v = F_U^{-1}(\pi(v), \lambda.f_U(v))$,where U is a trivializing neighbourhood at $\pi(v)$.

Let V be another trivializing neighbourhood at $\pi(v) = x$, then
$F_U \circ F_V^{-1}(x, \lambda.f_V(v)) = (x, g_{UV}(x).\lambda.f_V(v))$
$F_U \circ F_V^{-1}(x, \lambda.f_V(v)) = (x, g_{UV}(x).\lambda.g_{VU}(x)f_U(v)) = (x, \lambda.f_U(v))$
This implies that $F_V^{-1}(x, \lambda.f_V(v)) = F_U^{-1}(x, \lambda.f_U(v))$

Corollary(Framing property)-

Let $U \subset M$ be a trivializing neighbourhood. Then, for each $s \in \Gamma(U, L)$ there exists $s_U : U \to \mathbb{C}$ such that $s(x) = s_U(x).\xi_U(x)$, for each $x \in U$, where ξ_U is a nowhere vanishing section of U[4].

Proof-

Let $\xi_U \in \Gamma(U, L)$ be a framing section then by similar process as above,
$F_U(s(x)) = (x, \lambda(s(x).1)$
$s(x) = F_U^{-1}(x, \lambda(s(x)).1)$
Define $s_U : U \to \mathbb{C}$ by $s_U(x) = \lambda(s(x))$ By definition $s(x) = s_U(x).\xi(x)$

Framing sections under change of coordinates-

Let U and V be trivializing neighbourhoods with frames ξ_U and ξ_V. Let $x \in U \cap V$, then $F_V \circ F_U^{-1}(x, 1) = (x, g_{VU}(x))$ which implies that $F_U^{-1}(x, 1) = F_V^{-1}(x, g_{VU}(x))$ and $\xi_U(x) = g_{VU}(x)\xi_V(x)$

Proposition(Description via sections)-

Let (L, π) be a complex line bundle over M. Then $s : M \to L$ is a global section iff there exists open cover of trivializing neighbourhoods $\{U_\alpha | \alpha \in I\}$ and a family of functions $\{s_\alpha : U_\alpha \to \mathbb{C} | \alpha \in I\}$ such that $s_\alpha(x) = g_{\alpha\beta}(x)s_\beta(x)$, for every $x \in$

$U_\alpha \cap U_\beta$[4].

Proof-

i) Let $s \in \Gamma(M, L)$ be a global section and $\{U_\alpha | \alpha \in I\}$ be a trivializing cover of M. Then by above corollary for each U_α, $\exists s_\alpha : U_\alpha \to \mathbb{C}$ s.t. $s(x) = s_\alpha(x)\xi_\alpha(x)$ for all $x \in U_\alpha$.

Let $U_\alpha \cap U_\beta \neq \phi$, then $\xi_\alpha = g_{\beta\alpha}\xi_\beta$ on $U_\alpha \cap U_\beta$.
Since, s is globally defined $s_\alpha\xi_\alpha = s_\beta\xi_\beta = s_\beta(g_{\alpha\beta}\xi_\alpha)$. It follows that $s_\alpha = g_{\alpha\beta}s_\beta$

ii) Define $s|_{U_\alpha} = s_\alpha\xi_\alpha$ then $\pi \circ s|_{U_\alpha} = id_{U_\alpha}$
On $U_\alpha \cap U_\beta$,
$s|_{U\alpha} = s_\alpha\xi_\alpha = (g_{\alpha\beta}s_\beta)(g_{\beta\alpha}\xi_\beta) = s_\beta\xi_\beta = s|_{U_\beta}$
Hence, s is globally defined.

Chapter 4

Hormander's Theorem

4.1 Introduction

- The del-bar problem has been central to field of complex analysis since it captures holomorphic information about the complex plane separating it from the study of real structure on it. Hormander's approach that we would study is based on methodologies of complex Hilbert spaces- functional analytic approach, where the underlying notion would be captured by recognizing the function spaces as spaces of sections of a trivial line bundle equipped with a Hermitian metric.

4.2 Hormander's Theorem on the complex plane

Consider a complex-valued function $f \in C^\infty(\mathbb{C})$, then $\overline{\partial} f : \mathbb{C} \to T_{\mathbb{C}}^{*(0,1)}$ defined as $\dfrac{\partial f}{\partial \overline{z}} d\overline{z}$ is a $(0,1)$ form on $\mathbb{C}$. Let $\alpha \in \Gamma(\mathbb{C}, T_{\mathbb{C}}^{*(0,1)})$ be an arbitrary $(0,1)$ form then

$\alpha(z) = g(z)d\overline{z}$ for all $z \in \mathbb{C}$. We want to know when such a form can be written as $\alpha = \overline{\partial} f$, for some $f \in C^{\infty}(\mathbb{C})$. This is equivalent to solving the problem $\dfrac{\partial f}{\partial \overline{z}} = g$, for some function $f \in C^{\infty}(\mathbb{C})$.

The Hilbert space of solutions

The space of all smooth functions $C^{\infty}(\mathbb{C})$ under pointwise addition and scalar multiplication forms a $\mathbb{C}$-vector space [4]. To make it a inner product space we define the inner product $\langle , \rangle : C^{\infty}(\mathbb{C}) \times C^{\infty}(\mathbb{C}) \to \mathbb{C}$ with the weight $e^{-|z|^2}$ by the expression

$$\langle f, g \rangle = \frac{i}{2} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} dz \wedge d\overline{z}$$

Prptn- $\langle , \rangle : C^{\infty}(\mathbb{C}) \times C^{\infty}(\mathbb{C}) \to \mathbb{C}$ defined as above is an inner product.

Proof-

i) Bilinearity-

$\langle f_1 + f_2, g \rangle = \dfrac{i}{2} \displaystyle\int_{\mathbb{C}} (f_1 + f_2)\overline{g} e^{-|z|^2} dz \wedge d\overline{z}$

$= \dfrac{i}{2} \displaystyle\int_{\mathbb{C}} f_1 \overline{g} e^{-|z|^2} dz \wedge d\overline{z} + \dfrac{i}{2} \displaystyle\int_{\mathbb{C}} f_2 \overline{g} e^{-|z|^2} dz \wedge d\overline{z}$

$= \langle f_1, g \rangle + \langle f_2, g \rangle$

$$\langle f, \lambda g \rangle = \frac{i}{2} \int_{\mathbb{C}} f.\overline{\lambda g} e^{-|z|^2} dz \wedge d\overline{z} = \overline{\lambda} \langle f, g \rangle$$

ii) Non-negativity-

Let $f \in C^{\infty}(\mathbb{C})$, then $g(z) = |f(z)|^2 e^{-|z|^2}$ is a non-negative function on $\mathbb{C}$.

Taking integral on both sides, we get $\frac{i}{2} \displaystyle\int_{\mathbb{C}} g(z) dz \wedge d\overline{z} = \displaystyle\int_{R^2} g(x, y) dx dy \geq 0$

i.e. $\langle f, f \rangle \geq 0$

Furthermore, $\langle f, f\rangle = 0$ implies $|f(z)| = 0$ for all $z \in \mathbb{C}$ and hence $f = 0$.

iii) Conjugate Symmetry-

$\overline{\langle g, f\rangle} = \frac{-i}{2} \int\limits_{\mathbb{C}} \overline{g(z)} f(z) e^{-|z|^2} d\overline{z} \wedge dz = \frac{i}{2} \int\limits_{\mathbb{C}} \overline{g(z)} f(z) e^{-|z|^2} dz \wedge d\overline{z} = \langle f, g\rangle$

Define $V = \{f \in C^{\infty}(\mathbb{C})|$ such that $\langle f, f\rangle < \infty\}$.

Let $f, g \in V$ By Cauchy-Schwarz inequality $|\langle f, g\rangle|^2 \leq |\langle f, f\rangle|^2 . |\langle g, g\rangle|^2 < \infty$. Hence $\langle, \rangle$ is well-defined on V.

$\langle f + g, f + g\rangle = \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle < \infty$

$\langle \lambda f, \lambda f\rangle = |\lambda|^2 \langle f, f\rangle < \infty$ for all $\lambda \in \mathbb{C}$

This implies that $(V, \langle, \rangle)$ is an inner product space. The inner product space V can be equipped with topology induced by the L_2-norm given by $||.||_{L^2} = \langle, \rangle$. Denote $L^2(e^{-|z|^2}) = \overline{V}^{||.||_{L^2}}$, then the completion H is the Hilbert space wrt the L^2- norm $||.||$.

Such a choice of weight allows us to accomodate functions whose growth rate is lesser than the exponentials.

Densely defined Linear Operator(Definition)-

Let V, W be two topological vector spaces. We say T is a densely defined linear operator if $\exists$a dense subspace $D(T) \subset V$ s.t. $T : D(T) \to W$ is a linear operator.[3]

Adjoint Operator(Definition)-

Let V, W be two topological vector spaces and $T : D(T) \to W$ be a densely defined linear operator where $D(T) \subset V$ is a dense subspace. A densely defined operator $T^* : D(T^*) \to V$ is said to be adjoint of T if $\exists$ a dense subspace $D(T^*) \subset V$ such that

$\langle T\psi, \phi\rangle = \langle \psi, T^*\phi\rangle$ for all $\phi \in D(T^*)$ and $\psi \in D(T)$[3]

The del bar operator-

We can define $\overline{\partial} : C^\infty(\mathbb{C}) \to C^\infty(\mathbb{C})$ by $f \to \dfrac{\partial f}{\partial \overline{z}}$. Note that $\overline{\partial}$ is a $\mathbb{C}$-linear operator on the space $C^\infty(\mathbb{C})$.

$$\overline{\partial}(f+g) = \frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}} = \overline{\partial} f + \overline{\partial} g \text{ for all } f, g \in C^\infty\mathbb{C}$$

$$\overline{\partial}(cf) = \frac{\partial}{\partial \overline{z}}(cf) = c.\overline{\partial} f \text{ for all } c \in \mathbb{C} \text{ and } f \in C^\infty(\mathbb{C})$$

The space of compactly supported smooth functions $C_c^\infty(\mathbb{C})$ is dense in the Hilbert space $L^2(e^{-|z|^2})$. Hence, we see that the linear operator $\overline{\partial}$ is densely defined on $L^2(e^{-|z|^2})$. We construct its adjoint operator as follows[4]:

Let $f, g \in C_c^\infty(\mathbb{C})$

$$\langle g, \overline{\partial} f\rangle = \frac{i}{2} \int_{\mathbb{C}} g \overline{\frac{\partial f}{\partial \overline{z}}} e^{-|z|^2} dz \wedge d\overline{z} = \frac{i}{2} \int_{\mathbb{C}} g \frac{\partial \overline{f}}{\partial z} e^{-|z|^2} dz \wedge d\overline{z}$$

From integration by parts,
$= -\frac{i}{2}\int\limits_{\mathbb{C}} \overline{f}\frac{\partial}{\partial z}(ge^{-|z|^2})dz \wedge d\overline{z} = -\frac{i}{2}\int\limits_{\mathbb{C}} \overline{f}e^{|z|^2}\frac{\partial}{\partial z}(ge^{-|z|^2})e^{-|z|^2}dz \wedge d\overline{z}$

The adjoint operator of del bar(Definition)-
Define $\overline{\partial}^* : C_c^\infty(\mathbb{C}) \to C_c^\infty(\mathbb{C})$ by $\overline{\partial}^* g = -e^{|z|^2}\dfrac{\partial}{\partial z}(ge^{-|z|^2})$
We will get $\langle g, \overline{\partial} f\rangle = \langle \overline{\partial}^* g, f\rangle$ for all $f, g \in C_c^\infty(\mathbb{C})$

We define $\overline{\partial} : L^2(e^{-|z|^2}) \to L^2(e^{-|z|^2})$ by $f \to \overline{\partial} f$, where $\overline{\partial} f$ satisfies $\langle \overline{\partial} f, g\rangle = \langle f, \overline{\partial}^* g\rangle$ for all $g \in C_c^\infty(\mathbb{C})$

Statement- There exists $f \in L^2(e^{-|z|^2})$ such that $\overline{\partial} f \notin L^2(e^{-|z|^2})$ [4].

Consider the radial function $f(z) = \chi_{\mathbb{C}-B(0,1)}(z)$,then the L^2 norm of f is given by $\langle f, f\rangle = \dfrac{i}{2}\int\limits_{\mathbb{C}-B(0,1)} e^{-|z|^2}dz \wedge d\overline{z}$. In polar coordinates, we get

$$\langle f, f\rangle = \int\limits_{r=1}^{\infty}\int\limits_{\theta=0}^{2\pi} e^{-r^2} r dr d\theta = \pi\int\limits_{t=1}^{\infty} e^{-t}dt < \infty \implies f \in H$$

Let $\phi \in C_c^\infty(\mathbb{C})$,then
$\langle \overline{\partial} f, \phi\rangle = \langle f, \overline{\partial}^* \phi\rangle = -\frac{i}{2}\int\limits_{\mathbb{C}} f\frac{\partial}{\partial \overline{z}}(\overline{\phi}e^{-|z|^2})dz \wedge d\overline{z}$
Now for any smooth function g,

$$\frac{\partial g}{\partial \overline{z}} = \frac{\partial g}{\partial r}\frac{\partial r}{\partial \overline{z}} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial \overline{z}}$$

$r^2 = z\overline{z} \implies 2r\dfrac{\partial r}{\partial \overline{z}} = z \implies \dfrac{\partial r}{\partial \overline{z}} = \dfrac{e^{i\theta}}{2}$ and

$$e^{i2\theta} = \frac{z}{\overline{z}} \implies 2ie^{i2\theta}\frac{\partial\theta}{\partial\overline{z}} = \frac{-z}{\overline{z}^2} \implies \frac{\partial\theta}{\partial\overline{z}} = \frac{ie^{i\theta}}{2r}$$

$$\frac{\partial g}{\partial\overline{z}} = \frac{\partial g}{\partial r}\frac{e^{i\theta}}{2} + \frac{\partial g}{\partial\theta}\frac{ie^{i\theta}}{2r}$$

Choose $\phi(z) = e^{|z|^2}e^{i\theta}h(|z|)$, for some compactly supported smooth radial function h on $\mathbb{C}$

$$\begin{aligned}
\langle\overline{\partial}f, \phi\rangle &= -\frac{i}{2}\int\limits_{\mathbb{C}} f\frac{\partial}{\partial\overline{z}}(e^{-i\theta}h(|z|))dz \wedge d\overline{z} \\
&= -\frac{i}{2}\int\limits_{\mathbb{C}} f\left(e^{-i\theta}\frac{\partial h}{\partial r}\frac{e^{i\theta}}{2} + h(r)(-ie^{-i\theta})\frac{ie^{i\theta}}{2r}\right)dz \wedge d\overline{z} \\
&= -\frac{i}{2}\int\limits_{\mathbb{C}} \frac{f}{2}\left(\frac{\partial h}{\partial r} + \frac{h}{r}\right)dz \wedge d\overline{z}
\end{aligned}$$

Integrating via polar coordinates

$$= -\int\limits_{r=0}^{\infty}\int\limits_{\theta=0}^{2\pi}\frac{f}{2}\left(\frac{\partial h}{\partial r} + \frac{h}{r}\right)rdrd\theta = -\int\limits_{r=0}^{\infty}\int\limits_{\theta=0}^{2\pi}\frac{f}{2}\frac{\partial}{\partial r}(rh)drd\theta = -\pi\int\limits_{r=1}^{\infty}\frac{\partial}{\partial r}(rh)dr$$

We get that $\langle\overline{\partial}f, \phi\rangle = \pi h(1)$

We define a sequence of compactly supported smooth functions $\{\phi_\epsilon\}$ taking value π at $|z| = 1$ as follows: Let $\chi : [0, \infty) \to [0, 1]$ be a compactly supported smooth function then define $h_\epsilon(x) = \frac{1}{x}exp\left(\frac{-(x-1)^2}{\epsilon}\right)\chi(x)$

Now as $\epsilon \to 0$, $\phi_\epsilon \to 0$ uniformly

$$||\phi_\epsilon||^2 = \frac{i}{2}\int\limits_{\mathbb{C}} e^{2|z|^2}\frac{1}{|z|^2}e^{\frac{-2(|z|-1)^2}{\epsilon}}\chi(|z|)e^{-|z|^2}dz\wedge d\overline{z} = \int\limits_{r=\frac{1}{2}}^{2}\int\limits_{\theta=0}^{2\pi} e^{-\frac{2(r-1)^2}{\epsilon}}\frac{e^{r^2}}{r}drd\theta$$

$$=2\pi\int\limits_{r=\frac{1}{2}}^{2} e^{-\frac{2(r-1)^2}{\epsilon}}\frac{e^{r^2}}{r}dr \leq A.\int\limits_{r=\frac{1}{2}}^{2} e^{\frac{-2(r-1)^2}{\epsilon}}dr$$

This implies that $||\phi_\epsilon|| \to 0$ as $\epsilon \to 0$

By Cauchy-Schwarz inequality, $|\langle\overline{\partial}f, \phi\rangle|^2 \leq ||\overline{\partial}f||^2||\phi_\epsilon||^2$
$\pi^2|h(1)|^2 \leq ||\overline{\partial}f||^2||\phi_\epsilon||^2$

Since $\pi^2|h_\epsilon(1)|^2 = 1$ for all $\epsilon > 0$, this is a contradiction.

We define $D(\overline{\partial}) = \{f \in L^2(e^{-|z|^2}) | \overline{\partial}f \in L^2(e^{-|z|^2})\}$ as the domain of $\overline{\partial}$.
Let $f \in C_c^\infty(\mathbb{C})$, then f is zero except a compact set and $K = supp(\frac{\partial f}{\partial \overline{z}})$ is compact in $\mathbb{C}$. Hence, we get
$$\langle\frac{\partial f}{\partial \overline{z}}, \frac{\partial f}{\partial \overline{z}}\rangle = \frac{i}{2}\int\limits_K |\frac{\partial f}{\partial \overline{z}}|^2 e^{-|z|^2}dz\wedge d\overline{z} < \infty$$

It follows that $C_c^\infty(\mathbb{C}) \subset D(\overline{\partial}) \implies D(\overline{\partial})$ is dense in $L^2(e^{-|z|^2})$.

Hahn-Banach theorem for anti-linear functionals-

Suppose $T : W \to \mathbb{C}$ is an anti-linear functional, where $W \subset V$ is a subspace
Then we can write $T(v) = T_1(v) + iT_2(v)$
$T(iv) = -iT(v) \implies T_1(iv) + iT_2(iv) = -iT_1(v) + T_2(v)$
$T_1(v) = -T_2(iv)$ and $T_2(v) = T_1(iv)$
Hence $T(v) = T_1(v) + iT_1(iv)$

$$T_1(v) = \frac{T(v) + \overline{T(v)}}{2} \text{ For } r \in \mathbb{R}$$
$$T_1(v + r.w) = \frac{T(v + rw) + \overline{T(v + rw)}}{2} = \frac{T(v) + rT(w)}{2} + \frac{\overline{T(v) + rT(w)}}{2}$$
$$= \frac{T(v) + \overline{T(v)}}{2} + r.\frac{T(w) + \overline{T(w)}}{2} = T_1(v) + r.T_1(w)$$
This implies $T_1(v)$ is $\mathbb{R}$-linear.

By Hahn-Banach Theorem for $\mathbb{R}$-linear functionals, $\exists\ \tilde{T_1} : V \to \mathbb{R}$ such that $\tilde{T_1}|_W = T_1$. Define $\tilde{T} : V \to \mathbb{C}$ by $\tilde{T}(v) = \tilde{T_1}(v) + i\tilde{T_1}(iv)$, then $\tilde{T}$ is $\mathbb{R}$-linear such that $\tilde{T}|_W = T$
$\tilde{T}(iv) = \tilde{T_1}(iv) + i\tilde{T_1}(-v) = \tilde{T_1}(iv) - i\tilde{T_1}(v) = -i(i\tilde{T}(iv) + \tilde{T}(v)) = -i\tilde{T}(v)$ Since the extension is $\mathbb{C}$-linear, Hahn-Banach theorem works for anti-linear functionals.

L^2-Estimates of the $\overline{\partial}^*$ operator-

Let $f \in C_c^\infty(\mathbb{C})$,then[4]
$$\langle \overline{\partial}^* f, \overline{\partial}^* f \rangle = \langle \overline{\partial}(\overline{\partial}^* f), f \rangle = \frac{i}{2} \int\limits_{\mathbb{C}} \frac{\partial}{\partial \overline{z}} \left(-e^{|z|^2} \frac{\partial}{\partial z}(f(z)e^{-|z|^2}) \right) \overline{f} e^{-|z|^2} dz \wedge d\overline{z}$$

$$= -\frac{i}{2} \int\limits_{\mathbb{C}} \frac{\partial}{\partial \overline{z}} \left(e^{|z|^2} \left(\frac{\partial f}{\partial z} e^{-|z|^2} + f e^{-|z|^2}(-\overline{z}) \right) \right) \overline{f} e^{-|z|^2} dz \wedge d\overline{z}$$

$$= -\frac{i}{2}\int\limits_{\mathbb{C}} \frac{\partial}{\partial \overline{z}}\left(\frac{\partial f}{\partial z} - f\overline{z}\right)\overline{f}e^{-|z|^2}dz \wedge d\overline{z}$$

$$= -\frac{i}{2}\int\limits_{\mathbb{C}} \left(\frac{\partial^2 f}{\partial \overline{z}\partial z} - \frac{\partial f}{\partial \overline{z}}\overline{z} - f\right)\overline{f}e^{-|z|^2}dz \wedge d\overline{z}$$

$$= -\frac{i}{2}\int\limits_{\mathbb{C}} \left(\frac{\partial^2 f}{\partial z\partial \overline{z}} - \frac{\partial f}{\partial \overline{z}}\overline{z}\right)\overline{f}e^{-|z|^2}dz \wedge d\overline{z} + \frac{i}{2}\int\limits_{\mathbb{C}} |f|^2 e^{-|z|^2}dz \wedge d\overline{z}$$

$$= -\frac{i}{2}\int\limits_{\mathbb{C}} \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \overline{z}}e^{-|z|^2}\right)\overline{f}dz \wedge d\overline{z} + \frac{i}{2}\int\limits_{\mathbb{C}} |f|^2 e^{-|z|^2}dz \wedge d\overline{z}$$

By using integrtion by parts on $\mathbb{C}$,

$$= \frac{i}{2}\int\limits_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}e^{-|z|^2}\frac{\partial \overline{f}}{\partial z}dz \wedge d\overline{z} + \frac{i}{2}\int\limits_{\mathbb{C}} |f|^2 e^{-|z|^2}dz \wedge d\overline{z}$$

Now $\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$, on taking conjugate we get

$$\overline{\frac{\partial f}{\partial \overline{z}}} = \frac{1}{2}\left(\frac{\partial \overline{f}}{\partial x} - i\frac{\partial \overline{f}}{\partial y}\right) = \frac{\partial \overline{f}}{\partial z}$$

$$= \frac{i}{2}\int\limits_{\mathbb{C}} \left|\frac{\partial f}{\partial \overline{z}}\right|^2 e^{-|z|^2}dz \wedge d\overline{z} + \frac{i}{2}\int\limits_{\mathbb{C}} |f|^2 e^{-|z|^2}dz \wedge d\overline{z}$$

$$\langle \overline{\partial}^* f, \overline{\partial}^* f\rangle = \langle \overline{\partial} f, \overline{\partial} f\rangle + \langle f, f\rangle$$

Hormander's Theorem in $\mathbb{C}$-

Let $L^2(e^{-|z|^2})$ be the Hilbert space of complex valued functions defined as above and $\overline{\partial} : D(\overline{\partial}) \to L^2(e^{-|z|^2})$ be the del bar operator densely defined. Then, [4]
i)for all $g \in L^2(e^{-|z|^2})$ there exists $f \in D(\overline{\partial})$ such that $\overline{\partial} f = g$.
ii) Furthermore f, g satisfies the estimates
$\int\limits_{\mathbb{C}} |f|^2 e^{-|z|^2} dz \wedge d\overline{z} \leq \int\limits_{\mathbb{C}} |g|^2 e^{-|z|^2} dz \wedge d\overline{z}$

Proof- Consider the adjoint operator of del bar $\overline{\partial}^* : C_c^\infty(\mathbb{C}) \to C_c^\infty(\mathbb{C})$ then the space $\overline{\partial}^*(C_c^\infty(\mathbb{C}))$ is a linear subspace of $L^2(e^{-|z|^2})$.
Let $g \in L^2(e^{-|z|^2})$, we define $\lambda : \overline{\partial}^*(C_c^\infty(\mathbb{C})) \to \mathbb{C}$ by

$$\lambda(\overline{\partial}^* \phi) = \langle g, \phi \rangle$$

$\lambda(\xi . \overline{\partial}^* \phi) = \overline{\xi} . \lambda(\overline{\partial}^* \phi)$ for all $\xi \in \mathbb{C}$.
By Cauchy-schwarz inequality, $|\langle g, \phi \rangle| \leq ||g||.||\phi|| \leq ||g||.||\overline{\partial}^* \phi||$
i.e. $|\lambda(\overline{\partial}^* \phi)|^2 \leq ||g||.||\overline{\partial}^* \phi||$ This implies $\lambda : \overline{\partial}^*(C_c^\infty) \to \mathbb{C}$ is a bounded and anti-linear.

By Hahn-Banach theorem for anti-linear functionals $\exists\ \tilde{\lambda} : L^2(e^{-|z|^2}) \to \mathbb{C}$ such that $\tilde{\lambda}|_{\overline{\partial}^*(C_c^\infty(\mathbb{C}))} = \lambda$ and $||\lambda||_{(C_c^\infty)^*} = ||\tilde{\lambda}||_{L^2(e^{-|z|^2})}$

By Riesz- Representation theorem, there exists $f \in L^2(e^{-|z|^2})$ such that
for $\tilde{\lambda} : L^2(e^{-|z|^2}) \to \mathbb{C}$, $\tilde{\lambda}(h) = \langle f, h \rangle$ for all $h \in L^2(e^{-|z|^2})$ and $||\tilde{\lambda}||_{L^2(e^{-|z|^2})} = ||f||$
For all $\phi \in C_c^\infty(\mathbb{C})$ put $h = \overline{\partial}^* \phi$ and we get $\langle f, \overline{\partial}^* \phi \rangle = \tilde{\lambda}(\overline{\partial}^* \phi) = \lambda(\overline{\partial}^* \phi) = \langle g, \phi \rangle$

This implies $g = \dfrac{\partial f}{\partial \overline{z}}$.

Furthermore, $||\tilde{\lambda}|| = ||f|| \leq ||g||$ i.e. $\int\limits_{\mathbb{C}} |f|^2 e^{-|z|^2} dz \wedge d\overline{z} \leq \int\limits_{\mathbb{C}} |g|^2 e^{-|z|^2} dz \wedge d\overline{z}$.

Hence, the theorem allows us to solve the del bar equation and says that the derivatives will have the tendency to escape the solution space as evident from the example given in the Statement 1.

4.3 Hormander's Theorem on a Riemann surface

Consider a Riemann surface X and $H \to X$ be a holomorphic line bundle with a Hermitian metric $h = e^{-\phi}$ and we want to generalize the del-bar problem for X, for this purpose the $\overline{\partial}$ operator will be defined as a connection on space of smooth (0,1)-forms taking them to H-valued (0,1) forms.

Del-Bar Operator(Definition)-

Let $s \in \Gamma(X, H)$ be a smooth section, then $\exists$ collection of trivializing neigbourhoods $\{U_\alpha\}$ with frames $\{\xi_\alpha\}$ and transition functions $\{g_{\alpha\beta}\}$ such that $s|_{U\alpha} = f_\alpha \xi_\alpha$ where $f_\alpha : U_\alpha \to \mathbb{C}$ is smooth.

Define $\overline{\partial} s|_{U_\alpha} = \overline{\partial} f_\alpha \otimes \xi_\alpha$

On $U_\alpha \cap U_\beta$, $\xi_\alpha = g_{\beta\alpha}\xi_\beta$, and

we get $\overline{\partial} s|_{U_\alpha} = \overline{\partial} f_\alpha \otimes \xi_\alpha = \overline{\partial} f_\alpha \otimes g_{\beta\alpha}\xi_\beta = g_{\beta\alpha}\overline{\partial} f_\alpha \otimes \xi_\beta$

Since H is holomorphic ,$g_{\beta\alpha}$ is holomorphic $\overline{\partial} s|_{U_\alpha} = \overline{\partial}(g_{\beta\alpha} f_\alpha) \otimes \xi_\beta$

Now, $f_\beta \xi_\beta = f_\alpha \xi_\alpha = f_\alpha g_{\beta\alpha} \xi_\beta$ implying $f_\beta = f_\alpha g_{\beta\alpha}$

$\implies \overline{\partial} s|_{U_\alpha} = \overline{\partial} f_\beta \otimes \xi_\beta = \overline{\partial} s|_{U_\beta}$

This implies $\overline{\partial} : \Gamma(X, H) \to \Gamma(X, T_X^{*(0,1)} \otimes H)$ is well defined[4].

Holomorphic Line bundle(Definition)- Let H and X be complex manifolds and $\pi : H \to X$ be a map. We say the pair (π, H) is a holomophic line bundle iff:
i)(π, H) is a complex line bundle
ii)For each trivializing nbhd U, the trivializing map $F_U : \pi^{-1}(U) \to U \times \mathbb{C}$ is a holomorphic map.

Note that since F_U are diffeomorphism, by inverse function theorem they are biholomorpism as well. This implies that the transition data $g_{UV} : U \cap V \to \mathbb{C}^*$ are holomorphic.

Hermitian Metric(Definition)-
We say a smooth section $h : X \to H^* \otimes \overline{H^*}$ defined as $x \to h|_x$ is a Hermitian metric for the line bundle H if for all $x \in X$, $h|_x : H_x \otimes \overline{H_x} \to \mathbb{C}$ satisfies:
i)$h|_x(v, v) > 0$ and $h|_x(v, v) = 0$ iff $v = 0$ for all $v \in H_x$
ii)$h|_x(v + \lambda w, u) = h|_x(v, u) + \lambda h|_x(w, u)$ for all $v, w, u \in H_x$ and $\lambda \in \mathbb{C}$
iii)$h|_x(v, w) = \overline{h|_x(w, v)}$ for all $v, w \in H_x$

Integration of a (1,1)-form(Definition)-
Let X be a Riemann surface with a triangulation $\mathcal{T} = \{T_i | i \in I\}$ such that for each triangle $T_i \in \mathcal{T}$, $T_i \subset U_i$, where (U_i, z_i) is a trivializing neighbourhood and $\alpha \in \Gamma(X, \Lambda_X^{(1,1)})$ be a smooth (1,1) form on X. Then, $\alpha|_{U_i} = f_i dz_i \wedge d\overline{z_i}$ We define

$$\int\limits_X \alpha = \sum_{i \in I} \frac{i}{2} \int\limits_{z_i(T_i)} f_i(z_i) dz_i \wedge d\overline{z_i}$$

Here, for defining integration of (1,1)- forms the Riemann surface is assumed to be triangulated, however it is possible to give any Riemann surface a triangulation by solving Dirichlet problem on it[1].

Inner product structure on $\Gamma(X, H)$

Note that under pointwise addition and scalar multiplication both the spaces are $\mathbb{C}$-vector spaces[4]. We want to define an inner product structure for doing Hilbert space theory with del-bar operator.

Let $\omega \in \Gamma(X, \Lambda_X^{*(1,1)})$ be a positive (1,1) form on X and $s, t \in \Gamma(X, H)$ be smooth section. For each $x \in X$,$\exists U_x \subset X$ trivializing nbhd for H at $x \in X$ an U'_x trivializing for $T_X^{*(1,1)}$ at x. Therefore, $U = U_x \cap U'_x$ trivializes both H and $T_X^{*(1,1)}$ at $x \in X$. Therefore, we can say that there exists a collection of trivializing neighborhoods $\{U_\alpha\}$ such that $s = f_\alpha \xi_\alpha$ and $t = g_\alpha \xi_\alpha$ and $\omega = e^{-\eta_\alpha} dz_\alpha \wedge d\overline{z}_\alpha$ on U_α.

Consider a Hermitian metric $h : X \to H^* \otimes \overline{H^*}$ for the line bundle H.

For each $x \in X$,

$h|_x(s(x), t(x)) = h|_x(f_\alpha(x)\xi_\alpha(x), \overline{g_\alpha(x)\xi_\alpha(x)}) = f_\alpha(x)\overline{g_\alpha(x)}h(\xi_\alpha(x), \overline{\xi_\alpha(x)})$

Prptn- $h(s,t) : X \to \mathbb{C}$ is well-defined.

Proof- Let U_α and U_β be trivializing neighbourhoods such that $U_\alpha \cap U_\beta \neq \phi$.

On $U_\alpha \cap U_\beta$,

$h(s,t) = f_\alpha \overline{g_\alpha} h(\xi_\alpha, \overline{\xi_\alpha}) = f_\alpha \overline{g_\alpha} h(g'_{\beta\alpha}\xi_\beta, \overline{g'_{\beta\alpha}\xi_\beta})$

$h(s,t) = (g'_{\alpha\beta} f_\alpha)(\overline{g'_{\beta\alpha} g_\alpha}) h(\xi_\beta, \overline{\xi_\beta}) = f_\beta \overline{g_\beta} h(\xi_\beta, \overline{\xi_\beta})$

Thus, $h(s,t)$ is well-defined on X.

Define $s \wedge_\omega t = h(s,t)\omega$, where on each U_α, $s \wedge_\omega t = f_\alpha \overline{g_\alpha} e^{-\phi_\alpha} e^{-\eta_\alpha} dz_\alpha \wedge d\overline{z}_\alpha$ Since both $h(s,t)$ and ω are globally defined $s \wedge_\omega t$ is a global (1,1) form.

We define $\langle s,t \rangle = \int\limits_X s \wedge_\omega t$ for $s,t \in \Gamma(X,H)$.

Proposition- The map $\langle , \rangle : \Gamma(X,H) \times \Gamma(X,H) \to \mathbb{C}$ is an inner product and the space $(\Gamma(X,H), \langle , \rangle)$ is an inner product space.

Proof- i) $\langle s,s \rangle = \int\limits_X h(s,s)\omega$, Since $h(s,\overline{s}) \geq 0$ and $\omega \geq 0$ on X, $\langle s,s \rangle \geq 0$.

ii)$\overline{\langle s,t \rangle} = \int\limits_X \overline{h(s,\overline{t})\omega} = \int\limits_X h(t,\overline{s})\omega = \langle t,s \rangle$

Hence, $\langle \rangle$ is an inner product on $\Gamma(X,H)$.

Consider the space $V = \{s \in \Gamma(X,H) | \langle s,s \rangle < \infty\}$, then $V \subset \Gamma(X,H)$ is a vector subspace.

Proof- Let $f,g \in V$ then $|\langle f,g \rangle|^2 \leq ||f|||^2|g||^2 < \infty$

$\langle f+g, f+g \rangle = \langle f,f \rangle + 2Re(\langle f,g \rangle) + \langle g,g \rangle < \infty$

$\langle \lambda f, \lambda f \rangle = |\lambda|^2 \langle f,f \rangle < \infty$ for all $\lambda \in \mathbb{C}$

Hence, V is a vector subspace of $\Gamma(X,H)$.

Then, $L^2(\phi,\omega) = \overline{V}^{||\cdot||}$ is a Hilbert space with the norm induced by inner product.

Inner Product Structure on $\Gamma(X, T_X^{*(0,1)} \otimes H)$-

Let $\alpha, \beta \in \Gamma(X, T_X^{*(0,1)} \otimes H)$ be smooth sections of the H-valued (0,1) forms. For some collection of trivializing nbhds $\{U_\alpha\}$, we can write $\alpha = f_\alpha \xi_\alpha \otimes d\overline{z}_\alpha$ and $\beta = g_\alpha \xi_\alpha \otimes d\overline{z}_\alpha$ on each U_α. for some $f_\alpha, g_\alpha \in C^\infty(U_\alpha)$

Define $\dfrac{(\alpha \wedge \overline{\beta})e^{-\phi}}{2i} = \dfrac{-1}{2i} f_\alpha \overline{g}_\alpha e^{-\phi_\alpha} dz_\alpha \wedge d\overline{z}_\alpha = \dfrac{i}{2} f_\alpha \overline{g}_\alpha e^{-\phi_\alpha} dz_\alpha \wedge d\overline{z}_\alpha$ on U_α

Proposition- $\dfrac{\alpha \wedge \overline{\beta} e^{-\phi}}{2i}$ defined as above is a global (1,1) form[4].

Proof- Let us denote $g_{\beta\alpha}^{T_X^*(1,0)} = g_{\beta\alpha}^{dz}$ and $g_{\beta\alpha}^{T_X^*(0,1)} = g_{\beta\alpha}^{dz}$. We perform a change of coordinates from U_α to U_β on their intersection $U_\alpha \cap U_\beta$, we get that

$\xi_\alpha = g_{\beta\alpha}^H \xi_\beta$, $dz_\alpha = g_{\beta\alpha}^{dz} dz_\beta$ and $d\overline{z}_\alpha = g_{\beta\alpha}^{d\overline{z}} d\overline{z}_\beta$.

$$\frac{(\alpha \wedge \overline{\beta})e^{-\phi}}{2i} = \frac{i}{2} f_\alpha \overline{g}_\alpha h(g_{\beta\alpha}^H \xi_\beta, \overline{g_{\beta\alpha}^H \xi_\beta})(g_{\beta\alpha}^{dz} dz_\beta) \wedge (g_{\beta\alpha}^{d\overline{z}} d\overline{z}_\beta)$$

$$= \tfrac{i}{2}(g_{\beta\alpha}^H g_{\beta\alpha}^{dz} f_\alpha)\overline{(g_{\beta\alpha}^H g^{dz} g_\alpha)} h(\xi_\beta, \overline{\xi_\beta}) dz_\beta \wedge d\overline{z}_\beta$$

$$= \frac{i}{2} f_\beta \overline{g}_\beta e^{-\phi_\beta} dz_\beta \wedge d\overline{z}_\beta$$

Hence, $\dfrac{(\alpha \wedge \overline{\beta})e^{-\phi}}{2i}$ is well-defined.

We define $\langle \alpha, \beta \rangle = \dfrac{1}{2i} \displaystyle\int_X (\alpha \wedge \overline{\beta}) e^{-\phi}$ for $\alpha, \beta \in \Gamma(X, H \otimes T_X^{*(0,1)})$

Proposition- The map $\langle , \rangle : \Gamma(X, H \otimes T_X^{*(0,1)}) \times \Gamma(X, H \otimes T_X^{*(0,1)}) \to \mathbb{C}$ is an inner product.

Proof- i) Positivity- For $\alpha \in \Gamma(X, H \otimes T_X^{*(0,1)})$,

$\langle \alpha, \alpha \rangle = \sum_{T \in \mathcal{T}} \frac{i}{2} \int\limits_{z(T)} |f|^2 e^{-\phi} dz \wedge d\overline{z} \geq 0, \quad \text{where } \alpha|_{z(T)} = f\xi \otimes d\overline{z}$

ii)Conjugate Symmetry-

$\overline{\langle \alpha, \beta \rangle} = \sum_{T \in \mathcal{T}} \frac{-i}{2} \int\limits_{z(T)} \overline{f} g e^{-\phi} d\overline{z} \wedge dz = \sum_{T \in \mathcal{T}} \frac{i}{2} \int\limits_{z(T)} \overline{f} g e^{-\phi} dz \wedge d\overline{z} = \langle \beta, \alpha \rangle$

iii)Bilinearity $\langle \alpha_1 + \alpha_2, \beta \rangle = \sum_{T \in \mathcal{T}} \frac{i}{2} \int\limits_{z(T)} (f_1 + f_2) \overline{g} e^{-\phi} dz \wedge d\overline{z}$

$\langle \alpha_1 + \alpha_2, \beta \rangle = \sum_{T \in \mathcal{T}} \frac{i}{2} \int\limits_{z(T)} f_1 \overline{g} e^{-\phi} dz \wedge d\overline{z} + \sum_{T \in \mathcal{T}} \frac{i}{2} \int\limits_{z(T)} f_2 \overline{g} e^{-\phi} dz \wedge d\overline{z}$

$= \langle \alpha_1, \beta \rangle + \langle \alpha_2, \beta \rangle$

Define the space $W = \{\alpha \in \Gamma(X, H \otimes T_X^{*(0,1)}) | \langle \alpha, \alpha \rangle < \infty\}$. By similar calculations as before we can see that $(W, \langle , \rangle)$ is an inner product space. We can equip W with the metric topology induced by the inner product, we denote the completion by $\overline{W}^{||\cdot||} = L^2_{(0,1)}(\phi)$.

Adjoint operator of $\overline{\partial}$-

Define $\overline{\partial}^* : \Gamma_0(X, H \otimes T_X^{*(0,1)}) \to \Gamma_0(X, H)$ by $\beta \to \overline{\partial}^* \beta$ where $\overline{\partial}^* \beta$ satisfies $\langle \overline{\partial}^* \beta, s \rangle = \langle \beta, \overline{\partial} s \rangle$ for all $s \in \Gamma_0(X, H)$ [4]

Note that β is compactly supported, denote $K = supp(\beta)$, then K can be finitely triangulated which implies $\langle \beta, \overline{\partial} s \rangle = \int\limits_K \beta \wedge_\omega \overline{\overline{\partial} s} < \infty$

$$\langle \beta, \overline{\partial} s \rangle = \frac{1}{2i} \int\limits_K \beta \wedge \overline{\overline{\partial} s}$$

$$= \sum_{j=1}^{n} \frac{i}{2} \int\limits_{z(T_i)} f \overline{\frac{\partial h}{\partial \overline{z}}} e^{-\phi} dz \wedge d\overline{z} = \sum_{j=1}^{n} \frac{i}{2} \int\limits_{z(T_i)} f \frac{\partial \overline{h}}{\partial z} e^{-\phi} dz \wedge d\overline{z}$$

where $\beta|_{z(T_i)} = f \xi_i \otimes d\overline{z}_i$ and $s|_{z(T_i)} = h \xi_i$

Use integration by parts on each domain $z(T_i)$,

$$= \sum_{j=1}^{n} -\frac{i}{2} \int\limits_{z(T_i)} \frac{\partial}{\partial z}(f e^{-\phi}) \overline{h} dz \wedge d\overline{z} = \sum_{j=1}^{n} \frac{i}{2} \int\limits_{z(T_i)} -e^{\psi+\phi} \frac{\partial}{\partial z}(f e^{-\phi}) \overline{h} e^{-(\psi+\phi)} dz \wedge d\overline{z}$$

We define $\overline{\partial}^* \beta = -e^{\psi+\phi} \frac{\partial}{\partial z}(f e^{-\phi}) \xi_i = -e^{\psi} \left(\frac{\partial f}{\partial z} - f \frac{\partial \phi}{\partial z} \right) \xi_i$ on $z(T_i)$.Then,

$$\langle \overline{\partial}^* \beta, s \rangle = \sum_{j=1}^{n} \frac{i}{2} \int\limits_{z(T_i)} -e^{\psi+\phi} \frac{\partial}{\partial z}(f e^{-\phi}) \overline{h} e^{-(\psi+\phi)} dz \wedge d\overline{z} = \langle \beta, \overline{\partial} s \rangle$$

Statement: We claim that $\overline{\partial}^* \beta$ defined as above is globally defined.

Proof- Let U and V be a trivializing neighbourhood for H and $T_X^{*(0,1)}$ such that $U \cap V \neq \phi$ with $\overline{\partial}^* \beta|_U = -e^{\psi} \left(\frac{\partial f}{\partial z} - f \frac{\partial \phi}{\partial z} \right) \xi$ and $\overline{\partial}^* \beta|_V = -e^{\psi'} \left(\frac{\partial f'}{\partial z'} - f' \frac{\partial \phi'}{\partial z'} \right) \xi'$.

On $U \cap V$, we have relations, $e^{-\psi} = (g_{UV}^{dz} g_{UV}^{d\overline{z}})^{-1} e^{-\psi'}$, $e^{-\phi} = (g_{UV}^{H} \overline{g_{UV}^{H}}) e^{-\phi'}$ $\frac{\partial}{\partial z} = g_{UV}^{T_X^{(1,0)}} \frac{\partial}{\partial z'}$ and $\xi = g_{UV}^{H} \xi'$ and $f = g_{VU}^{H} g_{VU}^{d\overline{z}} f'$

We derive that $\frac{\partial f}{\partial z} = g_{UV}^{T_X^{(1,0)}} g_{VU}^{d\overline{z}} \frac{\partial}{\partial z'}(g_{VU}^{H} f') = g_{UV}^{T_X^{(1,0)}} g_{VU}^{d\overline{z}} \left(\frac{\partial g_{VU}^{H}}{\partial z'} f' + g_{VU}^{H} \frac{\partial f'}{\partial z'} \right)$

On differentiating $e^{-\psi} = (g_{UV}^{dz} g_{UV}^{d\overline{z}})^{-1} e^{-\psi'}$ we get

$$-e^{-\phi}\frac{\partial \phi}{\partial z} = g_{UV}^{T_X^{(1,0)}} \frac{\partial}{\partial z'}((g_{UV}^H \overline{g_{UV}^H}) e^{-\phi'}) = g_{UV}^{T_X^{(1,0)}} \overline{g_{UV}^H} \left(\frac{\partial g_{UV}^H}{\partial z'} e^{-\phi'} - g_{UV}^H e^{-\phi'} \frac{\partial \phi'}{\partial z'}\right)$$

$$g_{UV}^H \frac{\partial \phi}{\partial z} = g_{UV}^{T_X^{(1,0)}} \left(g_{UV}^H \frac{\partial \phi'}{\partial z'} - \frac{\partial g_{UV}^H}{\partial z'}\right)$$ This gives us,

$$\frac{\partial \phi}{\partial z} = (g_{UV}^H)^{-1} g_{UV}^{T_X^{(1,0)}} \left(g_{UV}^H \frac{\partial \phi'}{\partial z'} - \frac{\partial g_{UV}^H}{\partial z'}\right) = g_{UV}^{T_X^{(1,0)}} \left(\frac{\partial \phi'}{\partial z'} - \frac{1}{g_{UV}^H}\frac{\partial g_{UV}^H}{\partial z'}\right)$$

Putting it in $\left(\frac{\partial f}{\partial z} - f\frac{\partial \phi}{\partial z}\right)$, we get

$$= g_{UV}^{T_X^{(1,0)}} g_{VU}^{d\overline{z}} \left(\frac{\partial g_{VU}^H}{\partial z'} f' + g_{VU}^H \frac{\partial f'}{\partial z'} - (g_{VU}^H f') \left(\frac{\partial \phi'}{\partial z'} - \frac{1}{g_{UV}^H}\frac{\partial g_{UV}^H}{\partial z'}\right)\right)$$

$$= g_{VU}^{T_X^{(1,0)}} g_{VU}^{d\overline{z}} \left(\frac{\partial g_{VU}^H}{\partial z'} f' + g_{VU}^H \frac{\partial f'}{\partial z'} - g_{VU}^H f' \frac{\partial \phi'}{\partial z'} + \frac{1}{(g_{UV}^H)^2}\frac{\partial g_{UV}^H}{\partial z'} f'\right)$$

$$= g_{UV}^{T_X^{(1,0)}} g_{VU}^{d\overline{z}} g_{VU}^H \left(\left(\frac{\partial f'}{\partial z'} - f'\frac{\partial \phi'}{\partial z'}\right) + f'\left(g_{UV}^H \frac{\partial g_{VU}^H}{\partial z'} + g_{VU}^H \frac{\partial g_{UV}^H}{\partial z'}\right)\right)$$

$$= g_{UV}^{T_X^{(1,0)}} g_{VU}^{d\overline{z}} g_{VU}^H \left(\left(\frac{\partial f'}{\partial z'} - f'\frac{\partial \phi'}{\partial z'}\right) + f'\frac{\partial}{\partial z'}(g_{UV}^H g_{VU}^H)\right)$$

Since $g_{UV}^H g_{VU}^H = 1$, we get that

$$\left(\frac{\partial f}{\partial z} - f\frac{\partial \phi}{\partial z}\right) = g_{UV}^{T_X^{(1,0)}} g_{VU}^{d\overline{z}} g_{VU}^H \left(\frac{\partial f'}{\partial z'} - f'\frac{\partial \phi'}{\partial z'}\right)$$

So that $\overline{\partial}^* \beta|_U = ((g_{UV}^{dz} g_{UV}^{d\overline{z}}) e^{\psi'}) g_{UV}^{T_X^{(1,0)}} g_{VU}^{d\overline{z}} g_{VU}^H \left(\frac{\partial f'}{\partial z'} - f'\frac{\partial \phi'}{\partial z'}\right)(g_{UV}^H \xi')$

$$= e^{\psi'}\left(\frac{\partial f'}{\partial z'} - f'\frac{\partial \phi'}{\partial z'}\right)\xi'$$

Hence, $\overline{\partial}^* \beta$ is globally well-defined.

Del-bar Operator on $L^2(\phi,\omega)$-

Define $\overline{\partial} : L^2(\phi,\omega) \to L^2_{(0,1)}(\phi)$ by $s \to \overline{\partial}s$,
where $\overline{\partial}s$ satisfies $\langle \overline{\partial}s, \beta\rangle = \langle s, \overline{\partial}^*\beta\rangle$ for all $\beta \in \Gamma_0(X, H \otimes T_X^{*(0,1)})$ [4]

Define $Dom(\overline{\partial}) = \{s \in L^2(\phi,\omega) | \overline{\partial}s \in L^2_{(0,1)}(\phi)\}$. Since $\Gamma_0(X,H) \subset Dom(\overline{\partial})$, we see that $Dom(\overline{\partial})$ is a dense subspace of $L^2(\phi,\omega)$.

L^2- estimates on $\overline{\partial}^*$-

We have found that $\overline{\partial}^* : \Gamma_0(X, T_X^{*(0,1)} \otimes H) \to \Gamma_0(X,H)$ is given by
$\overline{\partial}^*\beta = -e^{\psi_\alpha+\eta_\alpha}\dfrac{\partial}{\partial z_\alpha}(e^{-\phi_\alpha}h_\alpha)\xi_\alpha$ on each U_α where $\beta = h_\alpha\xi_\alpha \otimes d\overline{z}_\alpha$ and U_α is a trivializing cover

Theorem(Bochner-Kodaira-Identity)-[4]
Let $\beta \in \Gamma_0(X, H \otimes T_X^{*(0,1)})$ be a compactly supported smoooth H-valued (0,1)-form.Let $\{U_\alpha\}$ be a open trivializing cover of $H \otimes T_X^{*(0,1)}$ and $\Lambda_X^{(1,1)}$. Then,

$$||\overline{\partial}^*\beta||^2 = \frac{1}{2i}\int\limits_X |\overline{\nabla}\beta|^2_\omega e^{-\phi}\omega + \frac{i}{2}\int\limits_X |\beta|^2_\omega e^{-\phi}\partial\overline{\partial}(\phi+\psi)$$

where if U is a trivializing neighbourhood of $H \otimes T_X^{*(0,1)}$ and $\Lambda_X^{(1,1)}$
$\beta|_U = f d\overline{z} \otimes \xi$, $\omega|_U = e^{-\psi} dz \wedge d\overline{z}$
$|\overline{\nabla}\beta|^2_\omega e^{-\phi} = e^{2\psi-\phi}|f_{\overline{z}} + \psi_{\overline{z}} f|^2$ and $|\beta|^2_\omega e^{-\phi}|_{U_\alpha} = e^{\psi-\phi}|f|^2$
are globally defined.

Proof-Let $\beta \in \Gamma_0(X, H \otimes T_X^{*(0,1)})$, then

$$\langle \overline{\partial}^* \beta, \overline{\partial}^* \beta \rangle = \langle \overline{\partial}(\overline{\partial}^* \beta), \beta \rangle = \frac{1}{2i} \int\limits_X \overline{\partial}(\overline{\partial}^* \beta) \wedge \overline{\beta} e^{-\phi}$$

Let U be a trivializing neighbourhood and $\beta = f d\overline{z} \otimes \xi$ on U.

$$\langle \overline{\partial}(\overline{\partial}^* \beta), \beta \rangle|_U = -\frac{i}{2} \int\limits_U \frac{\partial}{\partial \overline{z}} \left(e^{\psi} \left(\frac{\partial f}{\partial z} - f \frac{\partial \phi}{\partial z} \right) \right) \overline{f} e^{-\phi} dz \wedge d\overline{z}$$

$$= \frac{i}{2} \int\limits_U (-e^{\psi} \psi_{\overline{z}} (e^{-\phi} f)_z \overline{f} - e^{\psi - \phi} (f_{z\overline{z}} - \phi_z f_{\overline{z}}) \overline{f} + e^{\psi - \phi} \phi_{z\overline{z}} |f|^2) dz \wedge d\overline{z}$$

$$= \frac{i}{2} \int\limits_U (-e^{\psi} \psi_{\overline{z}} (e^{-\phi} f)_z \overline{f} - e^{\psi} (f_{\overline{z}} e^{-\phi})_z \overline{f} + e^{\psi - \phi} \phi_{z\overline{z}} |f|^2) dz \wedge d\overline{z}$$

Use integration by parts on U for 1st and 2nd term,

$$= \frac{i}{2} \int\limits_U (e^{\psi} \psi_{\overline{z}} \overline{f})_z e^{-\phi} f + (e^{\psi} \overline{f})_z f_{\overline{z}} e^{-\phi} + e^{\psi - \phi} \phi_{z\overline{z}} |f|^2) dz \wedge d\overline{z}$$

$$= \frac{i}{2} \int\limits_U (e^{\psi} \psi_z \psi_{\overline{z}} \overline{f} + e^{\psi} \psi_{z\overline{z}} \overline{f} + e^{\psi} \psi_{\overline{z}} (\overline{f})_z) e^{-\phi} f dz \wedge d\overline{z}$$

$$+ \frac{i}{2} \int\limits_U (e^{\psi} \psi_z \overline{f} + e^{\psi} (\overline{f})_z) f_{\overline{z}} e^{-\phi} dz \wedge d\overline{z} + \frac{i}{2} \int\limits_U e^{\psi - \phi} \phi_{z\overline{z}} |f|^2) dz \wedge d\overline{z}$$

$$= \frac{i}{2} \int\limits_U e^{\psi - \phi} (\psi_z \psi_{\overline{z}} \overline{f} + \psi_{\overline{z}} (\overline{f})_z f + \psi_z \overline{f} f_{\overline{z}} + (\overline{f})_z f_{\overline{z}}) dz \wedge d\overline{z}$$

$$+ \tfrac{i}{2} \int\limits_U e^{\psi - \phi} (\phi_{z\overline{z}} + \psi_{z\overline{z}}) |f|^2 dz \wedge d\overline{z}$$

$$= \frac{i}{2} \int\limits_U e^{2\psi - \phi} |\psi_z \overline{f} + f_{\overline{z}}|^2 e^{-\psi} dz \wedge d\overline{z} + \frac{i}{2} \int\limits_U e^{\psi - \phi} (\phi_{z\overline{z}} + \psi_{z\overline{z}}) |f|^2 dz \wedge d\overline{z}$$

ii) $\dfrac{\partial f}{\partial \overline{z}} = g_{VU}^{T_X^{(1,0)}} \dfrac{\partial}{\partial \overline{z'}}(g_{UV}^{d\overline{z}} g_{UV}^{H} f') = g_{VU}^{T_X^{(1,0)}} g_{UV}^{H} \left(\dfrac{\partial g_{UV}^{d\overline{z}}}{\partial \overline{z}'} f' + g_{UV}^{d\overline{z}} \dfrac{\partial f'}{\partial \overline{z}'} \right)$

Differentiating $e^{-\psi} = (g_{UV}^{dz} g_{UV}^{d\overline{z}}) e^{-\psi'}$ on both sides

$$-e^{-\psi} \frac{\partial \psi}{\partial \overline{z}} = g_{UV}^{T_X^{(1,0)}} \frac{\partial}{\partial \overline{z'}}((g_{UV}^{dz} g_{UV}^{d\overline{z}} e^{-\psi'}) = g_{UV}^{T_X(1,0)} g_{UV}^{dz} \left(\frac{\partial g_{UV}^{d\overline{z}}}{\partial \overline{z}'} e^{-\psi'} - e^{-\psi'} \frac{\partial \psi'}{\partial \overline{z'}} g_{UV}^{d\overline{z}} \right)$$

$$(g_{UV}^{dz} g_{UV}^{d\overline{z}}) e^{\psi'} \frac{\partial \psi}{\partial z} = g_{UV}^{T_X(1,0)} g_{UV}^{dz} \left(-\frac{\partial g_{UV}^{d\overline{z}}}{\partial \overline{z}'} e^{\psi'} + e^{\psi'} \frac{\partial \psi'}{\partial \overline{z'}} g_{UV}^{d\overline{z}} \right)$$

$$\frac{\partial \psi}{\partial z} = g_{UV}^{T_X(1,0)} \left(\frac{-1}{g_{UV}^{d\overline{z}}} \frac{\partial g_{UV}^{d\overline{z}}}{\partial \overline{z}'} + \frac{\partial \psi'}{\partial \overline{z'}} \right)$$

$$\frac{\partial f}{\partial \overline{z}} + f \frac{\partial \psi}{\partial \overline{z}} = g_{VU}^{T_X^{(1,0)}} g_{UV}^{H} \left(\left(\frac{\partial g_{UV}^{d\overline{z}}}{\partial \overline{z}'} f' + g_{UV}^{d\overline{z}} \frac{\partial f'}{\partial \overline{z}'} \right) + g_{UV}^{d\overline{z}} f' \left(\frac{-1}{g_{UV}^{d\overline{z}}} \frac{\partial g_{UV}^{d\overline{z}}}{\partial \overline{z}'} + \frac{\partial \psi'}{\partial \overline{z'}} \right) \right)$$

$$= g_{VU}^{T_X^{(1,0)}} g_{UV}^{H} g_{UV}^{d\overline{z}} \left(\frac{\partial f'}{\partial \overline{z'}} + f' \frac{\partial \psi'}{\partial \overline{z'}} \right)$$

So that $e^{2\psi-\phi}|\psi_z \overline{f} + f_{\overline{z}}|^2 = (g_{UV}^{dz} g_{UV}^{d\overline{z}})^{-2} |g_{VU}^{H}|^2 e^{2\psi'-\phi'} |g_{VU}^{T_X(1,0)} g_{UV}^{H} g_{UV}^{d\overline{z}}|^2 |\psi'_{\overline{z}} \overline{f'} + f'_{\overline{z'}}|$ Since $g_{VU}^{T_X(1,0)} = (g_{UV}^{dz})^{-1}$ we would get

$$e^{2\psi-\phi}|\psi_z \overline{f} + f_{\overline{z}}|^2 = e^{2\psi'-\phi'}|\psi'_z \overline{f'} + f'_{\overline{z'}}|$$

Hence, this expression is globally defined.

iii) $e^{\psi-\phi}|f|^2 = (g_{UV}^{dz} g_{UV}^{d\overline{z}} e^{-\psi'})^{-1} (g_{VU}^{H} \overline{g_{VU}^{H}} e^{-\phi'})(|g_{UV}^{H} g_{UV}^{d\overline{z}}|^2 f') = e^{\psi'-\phi'}|f'|^2$

iv) $\dfrac{\partial^2}{\partial z \partial \overline{z}}(.) dz \wedge d\overline{z} = |g_{VU}^{T_X^{(0,1)}}|^2 |g_{VU}^{T_X^{*(0,1)}}|^2 \dfrac{\partial^2}{\partial z' \partial \overline{z'}}(.) dz' \wedge d\overline{z'}$

Now, $e^{-(\psi+\phi)} = |g_{UV}^{H}|^2 |g_{UV}^{dz}| e^{-(\psi'+\phi')}$ Since, g_{UV}^{H} and g_{UV}^{dz} are holomorphic their modulus is a harmonic function on $U \cap V$ putting it above we get,

$$\frac{\partial^2}{\partial z \partial \overline{z}}(e^{-(\psi+\phi)}) dz \wedge d\overline{z} = |g_{UV}^{H}|^2 |g_{UV}^{dz}|^2 \frac{\partial^2}{\partial z' \partial \overline{z'}}(e^{-(\psi'+\phi')}) dz' \wedge d\overline{z'}$$

$$-e^{-(\psi+\phi)}\frac{\partial^2}{\partial z\partial\overline{z}}(\psi+\phi)dz\wedge d\overline{z} = -|g_{UV}^{H}|^2|g_{UV}^{dz}|^2e^{-(\psi'+\phi')}\frac{\partial^2}{\partial z'\partial\overline{z'}}(\psi'+\phi')dz'\wedge d\overline{z'}$$

Hence, $\partial\overline{\partial}(\phi+\psi)$ is globally defined (1,1) form.

Hormander's Theorem-[4]

Let $H \rightarrow X$ be a holomorphic line bundle over a Riemann surface X. Let H be equipped with a hermitian metric h and $\omega \in \Gamma(X, \Lambda_X^{(1,1)})$ be a positive $(1,1)$ form on X.If $\partial\overline{\partial}(\phi+\psi) \geq c\omega$, for some $c > 0$, then

i) for every $\alpha \in L^2_{(0,1)}(\phi)$ $\exists$ $u \in L^2(\phi,\omega)$ s.t. $\overline{\partial}u = \alpha$

ii) $\displaystyle\int_X |u|^2e^{-\phi}\omega \leq \frac{1}{c}\left(\frac{1}{2i}\int\limits_X \alpha\wedge\overline{\alpha}e^{-\phi}\right)$

Proof-Let $\alpha \in L^2_{(0,1)}(\phi)$ be a H-valued $(0,1)$ form.

Consider the adjoint operator $\overline{\partial}^* : \Gamma_0(X, H\otimes T_X^{*(0,1)}) \rightarrow \Gamma_0(X,H)$, then since $\overline{\partial}^*$ is a linear operator $\overline{\partial}^*(\Gamma_0(X,H\otimes T_X^{*(0,1)}))$ is a linear subspace of $L^2(\phi,\omega)$.
We define $\lambda : \overline{\partial}^*(\Gamma_0(X,H\otimes T_X^{*(0,1)})) \rightarrow \mathbb{C}$ by $\lambda(\overline{\partial}^*\beta) = \langle\alpha,\beta\rangle$.

Note that for $\beta,\gamma \in \Gamma_0(X,H\otimes T_X^{*(0,1)})$ and $c \in \mathbb{C}$
$\lambda(\overline{\partial}^*(\beta+c\gamma)) = \langle\alpha,\beta+c\gamma\rangle = \langle\alpha,\beta\rangle + \overline{c}\langle\alpha,\gamma\rangle = \lambda(\overline{\partial}^*\beta) + \overline{c}\lambda(\overline{\partial}^*\gamma)$

By Cauchy-Schwarz theorem, $|\langle\alpha,\beta\rangle|^2 \leq \langle\alpha,\alpha\rangle\langle\beta,\beta\rangle$

By previous identity,

$$||\overline{\partial}^*\beta||^2 \geq \frac{i}{2}\int\limits_X |\beta|^2_\omega e^{-\phi}\partial\overline{\partial}(\phi+\psi) \geq c\left(\frac{i}{2}\int\limits_X |\beta|^2_\omega e^{-\phi}\omega\right) = c.||\beta||^2$$

This implies that $|\lambda(\overline{\partial}^*\beta)|^2 \leq \dfrac{1}{c}||\alpha||^2||\overline{\partial}^*\beta||^2$i.e. λ is a continuous anti-linear functional.

By Hahn-Banach theorem for continuous anti-linear functional, $\exists\ \widetilde{\lambda} : L^2(\phi, \omega) \to \mathbb{C}$ such that i) $\widetilde{\lambda}|_{\Gamma_0(X, H\otimes T_X^{*(0,1)})} = \lambda$ and ii) $||\lambda|| = ||\widetilde{\lambda}||$

Since $L^2(\phi, \omega)$ is a Hilbert space, By Riesz-Representation theorem $\exists\, u \in L^2(\phi, \omega)$ such that $\widetilde{\lambda}(s) = \langle u, s\rangle$ for all $s \in L^2(\phi, \omega)$ and $||\widetilde{\lambda}|| = ||u||$.

In particular for all $\beta \in \Gamma_0(X, H \otimes T_X^{*(0,1)})$, $\widetilde{\lambda}(\overline{\partial}^*\beta) = \langle u, \overline{\partial}^*\beta\rangle$

By definition $\langle \alpha, \beta\rangle = \lambda(\overline{\partial}^*\beta) = \widetilde{\lambda}(\overline{\partial}^*\beta) = \langle u, \overline{\partial}^*\beta\rangle$

This implies that $\overline{\partial}u = \alpha$ in the sense of currents as defined.

Furthermore, we have that $||u|| = ||\lambda|| = ||\widetilde{\lambda}|| \leq \frac{1}{\sqrt{c}}||\alpha||$.

4.4 Conclusion

The following report gives an idea about how rich the theory of Riemann surfaces can be.They serve as a fundamental building block for learning about the classical theory of manifolds (smooth or topological) which further evolves into methods of modern geometry through line bundles or more stringently holomorphic line bundles. Modern techniques of pde from functional analysis to operator theroy can be studied on the space of sections of operators defined on manifolds out of purely geometric motivations. The solution of ∂ and $\partial\overline{\partial}$ is essential in studying the phenomenon of analytic continuation which are well-known for Riemann surfaces[4]. Further applications of Hormander's theorem lies in the embedding problem of Riemann surfaces which allows us to treat them as domains in some $\mathbb{C}^n$,$n > 1$ [4].

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