On joins and construction of $K(G, 1)$ spaces

A thesis submitted in partial fulfillment of the
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by

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Under the supervision of
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to the

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Declaration

I hereby declare that this thesis is my own work and to the best of my knowledge, it contains no materials previously published or written by any other person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at IISER Kolkata or any other educational institution, except where due acknowledgement is made in the thesis.

April 2017
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Piduri Chandrahas

Certificate

This is to certify that the thesis entitled "On Joins and Construction of $K(G, 1)$ spaces" is a *bona fide* record of work done by Piduri Chandrahas (12MS083), a student enrolled in BS-MS Dual Degree Programme, under my supervision during August 2016 - April 2017, submitted in partial fulfillment of the requirements for the award of BS-MS Dual Degree to the Department of Mathematics and Statistics (DMS), Indian Institute of Science Education and Research (IISER) Kolkata.

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Piduri Chandrahas
IISER Kolkata
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To my parents

Abstract

The principal aim of this thesis is to construct $K(G, 1)$ spaces for any given group G with discrete topology. The general construction of universal G-bundles and classifying spaces by Milnor is used to acheive this. Uniqueness of $K(G, 1)$ spaces is established for a particular class of groups G. Milnor's construction relies on the join of spaces. A major theme of this thesis is to compare various topological joins. We extend the notion of joins for an arbitrary family of spaces.

Contents

Preface

A path connected space whose only non-trivial homotopy group is its n^{th} homotopy group π_n is called a $K(G,n)$ space, where G is a group isomorphic to π_n. These were introduced and studied in [Eilenberg and MacLane, 1945] and [Eilenberg and MacLane, 1950]. This dissertation studies a construction of a $K(G,1)$ space for a group G with discrete topology. For this, a more general construction (see [Milnor, 1956b]) of universal G-bundles and classifying spaces of groups. These classifying spaces are built using the concept of the join of spaces.

A particular kind of topology is defined on the join of spaces that enables Milnor's construction of classifying spaces. However, one can define other topologies too on the join of spaces. A major theme of this thesis is to compare the various topologies on the join of arbitrary family of spaces. We examine if a construction of $K(G,1)$ spaces is possible using Milnor's construction with these other topologies.

In chapter 1, graphs are considered as topological spaces; their fundamental groups and covering spaces are discussed. It is proved that the fundamental group of a graph is a free group. Using covering space theory, various algebraic properties of a free group and its (normal) subgroups are realized geometrically; for instance, every subgroup of a free group is free. For a given free group G with discrete topology, one obtains a graph to be a $K(G,1)$ space.

Chapter 2 gives an introduction to CW-complexes. In particular, we examine the CW-complex structure on the infinite sphere and compare it with other topologies on the infinite sphere. We look at the group action of the unit circle on the infinite sphere.

Chapter 3 discusses joins of spaces. The join of two spaces is defined in several ways: as a space of line segments, as a quotient space, and as a space of formal convex combinations. These topological joins are compared. We extend the notion of the join to arbitrary family of spaces. We examine a case when these joins are equivalent; this case will be useful in chapter 5.

Chapter 4 is a superficial introduction to the theory of fiber bundles and principal G-bundles. Only the concepts required for chapter 5 are described.

In Chapter 5, Milnor's construction of universal G-bundles and classifying spaces of a topological group G are discussed. By taking G to be a group with disrcete topology, the base space of the universal G-bundle is obtained to be a $K(G,1)$ space. Uniqueness of a $K(G,1)$ space, up to homotopy type, is guaranteed if the $K(G,1)$ space is a CW-complex. Hence, a CW-complex structure is described for the $K(G,1)$ space obtained from Milnor's construction. Finally, examples of $K(G,1)$ spaces are considered.

There are simplicial methods ([Hatcher, 2002] p.89) for construction of $K(G,1)$ spaces. Milnor's construction, however, is a more general construction. It shows the existence of a classifying space BG of a given topological group G. A classifying space BG of a group G is the base space of a universal G-bundle. The assignment $G \mapsto BG$ is a functor from the category of topological groups to the category of topological spaces. The classifying space BG is primarily important because there is a bijection between the homotopy classes of maps $X \to BG$ and isomorphism classes of principal G-bundles over a paracompact Hausdorff space X.

In spite of best efforts of the author, there might be some errors of both typographical and mathematical in nature. The author is solely responsible for such errors.

Notations

$A \subset B$: inclusion of sets, not necessarily proper
$A \backslash B$: the set of elements in A but not in B
$A \cup B$: union of sets A and B
$A \cap B$: intersection of sets A and B
$\mathbb{N}$: the set of natural numbers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{Z}$: the set of integers
$\mathbb{Q}$: the set of rational numbers
$\mathbb{R}$: the set of real numbers
$\mathbb{C}$: the set of complex numbers
$\mathbb{Z}_n$: the set of integers modulo n
$\mathbb{R}^n$: the n-dimensional euclidean space, where n is a positive integer
$\mathbb{C}^n$: the n-dimensional complex space, where n is a positive integer
S^n : the unit sphere in $\mathbb{R}^{n+1}$
D^n : the unit disk or ball in $\mathbb{R}^n$
I : the closed unit interval $[0, 1]$
$\{*\}$: the one-point space
$\coprod$: disjoint union of sets or spaces
$\times, \prod$: product of sets or spaces
$\bar{A}$: the closure of the (sub)space A
A° : the interior of the (sub)space A
pr_A : the projection map onto A

Chapter 1

Graphs and Free Groups

Graphs have been traditionally studied in combinatorics. In this chapter, graphs will be considered as topological spaces, thus enabling one to talk about their fundamental groups. The exposition here largely follows [Hatcher, 2002](p. 83-87).

1.1 Graphs and trees

This section shows the existence of a maximal tree in a connected graph. The computation of the fundamental group of a graph relies on the existence of a maximal tree in the graph.

Definition 1.1. *Let X^0 be a discrete set and $\{I_\alpha\}_{\alpha\in\Lambda}$ be an indexed collection of unit closed intervals. Consider the disjoint union $X^0 \coprod_\alpha I_\alpha$ with disjoint union topology, and family of maps $\{\phi_\alpha : \partial I_\alpha \to X^0\}_\alpha$. The quotient space X obtained from $X^0 \coprod_\alpha I_\alpha$ by the identifications $x \sim \phi_\alpha(x)$ for $x \in \partial I_\alpha$ and $\alpha \in \Lambda$ is called a **graph** .*

Example 1.2. Consider a singleton $\{x_0\}$ and let $\{I_\alpha\}_{\alpha\in\Lambda}$ be an indexed collection of unit closed intervals. The graph X obtained by the maps $\{\phi_\alpha : \partial I_\alpha \to \{x_0\}\}_\alpha$ is called a wedge sum of circles indexed over Λ with base point $x_0 \in X$. It is denoted by $\vee_{\alpha\in\Lambda} S^1_\alpha$. When Λ is a finite set of cardinality n, we simply call X as a wedge of n circles. Refer figure 1.1.

Denote the quotient map $X^0 \coprod_\alpha I_\alpha \to X$ sending each point to its equivalence class under the identifications of definition 1.1 by q.

Definition 1.3. *The points in X^0 are called the **vertices** of the graph X .*

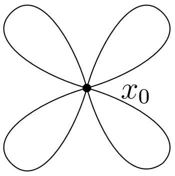


Figure 1.1: A wedge of four circles.

Definition 1.4. *For $\alpha \in \Lambda$, the image of $I_\alpha \backslash \partial I_\alpha$ under the quotient map q is called an* **edge** *(denoted bye_α) of the graph X.*

Without being too pedantic, we might refer to the images of points in X^0 under q, too, as vertices. Two vertices are said to be **adjacent** if there is an edge e_α such that end points of I_α are identified with these two vertices respectively. An edge e_α is said to be **incident** on a vertex if one of the endpoints of I_α is identified with this vertex.

Lemma 1.5. *A graph is a Hausdorff topological space.*

Proof. Let p_1 and p_2 be any two points of a graph X. Let the collection of edges incident on p_1 be $\{e_\alpha^{p_1}\}_\alpha$ and the collection of edges incident on p_2 be $\{e_\beta^{p_2}\}_\beta$. Also, let the collection of edges joining p_1 and p_2 be $\{e_\gamma^{p_1 p_2}\}_\gamma$. These collections of edges are not necessarily non-empty. We have the following cases.

(i) The points p_1 and p_2 belong to distinct edges e_α and e_β respectively.
Then the edges e_α and e_β are open sets in X that separate p_1 and p_2 respectively.

(ii) Both p_1 and p_2 are in the same edge e_α.
Separate pre-images of p_1 and p_2 in $I_\alpha \backslash \partial I_\alpha$ using two open sets in I respectively. Then the images of these two open sets under q in e_α are open sets that separate p_1 and p_2 respectively.

(iii) The points p_1 and p_2 are not adjacent vertices.
Then the open sets $\{e_\alpha^{p_1}\}_\alpha \cup \{p_1\}$ and $\{e_\beta^{p_2}\}_\beta \cup \{p_2\}$ separate p_1 and p_2 respectively.

(iv) The points p_1 and p_2 are adjacent vertices.
For each γ, let U_γ and V_γ be the images of the open sets separating p_1 and p_2 in I_γ respectively. Then, $(\{e_\alpha^{p_1}\}_\alpha \backslash \{e_\gamma^{p_1 p_2}\}_\gamma) \cup \{U_\gamma\}_\gamma$ and $(\{e_\beta^{p_2}\}_\beta \backslash \{e_\gamma^{p_1 p_2}\}_\gamma) \cup \{V_\gamma\}_\gamma$ are open sets in X separating p_1 and p_2 respectively.

(v) The point p_1 is in e_α and p_2 is a vertex such that e_α is not incident on p_2. Then e_α and $\{e_\beta^{p_2}\}_\beta \cup \{p_2\}$ are open sets that separate p_1 and p_2 respectively.

(vi) The point p_1 is in e_α and p_2 is a vertex such that e_α is incident on p_2. Let the images of open sets separating $q^{-1}(p_1)$ and $q^{-1}(p_2)$ in I_α be U_1 and U_2 respectively. Then U_1 and $(\{e_\beta^{p_2}\}_\beta \backslash \{e_\alpha\}) \cup \{U_2\}$ are open sets in X that separate p_1 and p_2.

■

Each edge is homeomorphic to the open unit interval. We also have the following.

Lemma 1.6. *The closure of an edge is homeomorphic to the unit closed interval or the unit circle.*

Proof. Consider the continuous map Φ_α associated with an edge e_α defined as the composition $I_\alpha \hookrightarrow X^0 \coprod_\alpha I_\alpha \xrightarrow{q} X$. We see that $\Phi_\alpha|_{\partial I_\alpha} = \phi_\alpha$. Also, $\Phi_\alpha|_{\operatorname{int} I_\alpha} : \operatorname{int} I_\alpha \to e_\alpha$ is a homeomorphism. Hence $e_\alpha = \Phi_\alpha(\operatorname{int} I_\alpha) \subset \Phi_\alpha(I_\alpha) \subset \overline{\Phi_\alpha(\operatorname{int} I_\alpha)} = \bar{e}_\alpha$ where the second inclusion follows from the continuity of Φ_α. But $\Phi_\alpha(I_\alpha)$ is compact in the Hausdorff space X whence $\Phi_\alpha(I_\alpha) = \bar{e}_\alpha$. Therefore e_α is homeomorphic to S^1 if $\phi_\alpha(\partial I_\alpha)$ is a singleton, otherwise it is homeomorphic to I. ■

Definition 1.7. *Let X be a graph. Define a topology on X by declaring a subset of X to be open (or closed) if and only if it intersects the closure $\bar{e}_\alpha$ of every edge e_α in an open (or closed) set of $\bar{e}_\alpha$. This topology is called the* ***weak topology*** *of graph X with respect to the subspaces $\bar{e}_\alpha$.*

Lemma 1.8. *Quotient topology of a graph is equivalent to its weak topology with respect to the closures of edges.*

Proof. Let X be a graph. If $A \subset X$ is in the quotient topology, then A is in the weak topology. Now let $A \subset X$ be in the weak topology. We have to show that $q^{-1}(A) \cap I_\alpha$ is open for each α. Define the continuous map Φ_α by the composition $I_\alpha \hookrightarrow X^0 \coprod_\alpha I_\alpha \to X$. We have the following commutative diagram.

$$\begin{tikzcd}
I_\alpha \arrow[r, hook, "\iota_\alpha"] \arrow[dr, "\Phi_\alpha"'] & X^0 \coprod_\alpha I_\alpha \arrow[d, "q"] \\
& \bar{e}_\alpha
\end{tikzcd}$$

Thus $\Phi_\alpha^{-1}(A \cap \bar{e}_\alpha)$ is open in I_α, which implies that $\iota_\alpha^{-1} \circ q^{-1}(A \cap I_\alpha)$ is open in I_α. This gives our result. ■

The proof of the next lemma is clear.

Lemma 1.9. *Consider a graph and the collection of open sets in edges and path connected neighborhoods of vertices. Then this collection forms a basis for the weak topology of the graph with respect to the closures of edges.* ■

Corollary 1.10. *A graph is connected if and only if it is path connected.*

Proof. Each element of the basis defined in the above lemma is path connected. ■

Definition 1.11. *A subspace Y of a graph is called a* ***subgraph*** *if it consists of vertices and edges such that the closure of an edge $e \subset Y$ is in Y.*

A subgraph is a closed subspace of a graph. This means that a subgraph too has weak topology with respect to the closures of edges contained in the subgraph. Hence a subgraph is a graph.

Definition 1.12. *A path connected (sub)graph that is contractible is called a* ***tree****.*

Definition 1.13. *A tree in a graph is called a* ***maximal tree*** *if the tree contains all the vertices of the graph.*

Theorem 1.14. *Every connected graph contains a maximal tree.*

Proof. Let X be a connected graph that has the weak topology with respect to the closures of edges e_α. We shall prove that if a subspace X_0 of X is given, then X_0 can be embedded in a subspace Y of X that contains all the vertices of X and deformation retracts to X_0. The theorem is then proved by setting X_0 to be a vertex of X.

Step 1 Consider X_0. Construct $X_1 \subset X$ by adding all the closures $\bar{e}_\alpha$ that have at least one endpoint in X_0. Inductively construct X_{i+1} from X_i , for each non-negative integer i, by adding all $\bar{e}_\alpha$ with at least one endpoint in X_i. We see that $X_0 \subset \ldots \subset X_i \subset X_{i+1} \subset \ldots$ is a sequence of subgraphs. Let x be a point in $\cup_{i\in\mathbb{N}_0} X_i$. If $x \in X_i$, then by construction there exists an open neighborhood of x that is contained in X_{i+1}. Therefore $\cup_{i\in\mathbb{N}_0} X_i$ is open in X. Also, since $\cup_{i\in\mathbb{N}_0} X_i$ is a union of closures of edges, it is closed in X. Hence $\cup_{i\in\mathbb{N}_0} X_i = X$.

Step 2 Now set $Y_0 = X_0$. We construct Y_{i+1} inductively from Y_i. For each vertex v of $X_{i+1} \backslash X_i$, consider an edge connecting v to Y_i. Obtain Y_{i+1} by adjoining all such edges to Y_i. The space Y_{i+1} deformation retracts to Y_i because the edges adjoined to Y_i deformation retract to their endpoints in Y_i. Denote this deformation retraction $I \times Y_{i+1} \to Y_{i+1}$ as h_i for $i \in \mathbb{N}_0$.

Step 3 Setting $Y = \cup_{i \in \mathbb{N}_0} Y_i$, define a homotopy $h : I \times Y \to Y$ by

$$h(t,y) = \begin{cases} h_i(t,y), & \text{if } y \in Y_{i+1} \text{ and } \quad t \in [2^{-i-1}, 2^{-i}] \\ y, & \text{otherwise}. \end{cases}$$

Let $A \subset Y$ be open. So $A \cap \bar{e}_\alpha$ is open for each α. If $\bar{e}_\alpha \subset Y_{i+1}$ then $h^{-1}(A \cap \bar{e}_\alpha) = h_i^{-1}(A \cap \bar{e}_\alpha)$. Since h_i is continuous, it follows that h is continuous. ■

1.2 Fundamental group and coverings of graphs

In this section, it will be shown that the fundamental group of a graph is a free group and that every covering space of a graph is a graph.

Definition 1.15. *Let X be a connected graph with a maximal tree T. Let e_α be an edge in X. With base point $x_0 \in T$, construct a loop γ_α at x_0 by traveling along a path in T joining x_0 to an endpoint of e_α followed by the edge e_α and then continuing along a path in T joining the other endpoint to x_0. The path γ_α is said to be a* ***loop determined by the edge*** *e_α at the base point x_0. The path homotopy class $[\gamma_\alpha]$ is said to be the* ***loop class determined by the edge*** *e_α at the base point x_0.*

Strictly speaking, the edge e_α must be first oriented to determine the corresponding loop class. This, however, will not make a difference because a group containing these loop classes will be considered.

Since T is contractible, the loop class determined by an edge in T is the trivial class of loops based at x_0. Each edge e_α in $X \backslash T$ determines a loop class $[\gamma_\alpha]$ that is independent of the choice of paths joining x_0 to the endpoints of e_α.

The following definition and lemmas are quoted from [Hatcher, 2002](p. 14-16).

Definition 1.16. *Let X and Y be topological spaces and let $T \subset X$. Further let a continuous map $f_0 : X \to Y$ and a homotopy $f : T \times I \to Y$ be given such that $f\,|_{T \times \{0\}} = f_0\,|_T$. If $f : T \times I \to Y$ can be extended to a homotopy*

$f : X \times I \to T$ such that $f|_{X\times\{0\}} = f_0$, then the pair (X, T) is said to satisfy ***homotopy extension property****.*

Lemma 1.17. *Let X be a connected graph with a maximal tree T. Then the pair (X, T) satisfies the homotopy extension property.*

Lemma 1.18. *If a pair (X, T) satisfies the homotopy extension property and T is contractible, then the quotient map $X \to {}^{X}\!/_{T}$ is a homotopy equivalence.*

Now the fundamental group of a graph can be computed.

Theorem 1.19. *Let X be a connected graph with a maximal tree T. Then the fundamental group $\pi_1(X, x_0)$ is a free group where the base point $x_0 \in T$. A basis is given by the loop classes determined by the edges in $X\backslash T$ at the base point x_0.*

Proof. From lemmas 1.17 and 1.18, the quotient map $q' : X \to {}^{X}\!/_{T}$ obtained by collapsing T is a homotopy equivalence. Since composition of quotient maps is a quotient map, the quotient space ${}^{X}\!/_{T}$ is also a graph. Because T contains all the vertices of X, the graph ${}^{X}\!/_{T}$ contains only one vertex. The edges in ${}^{X}\!/_{T}$ correspond to the edges in $X\backslash T$. Therefore ${}^{X}\!/_{T}$ is a wedge sum of circles indexed over the loop classes $[\gamma_\alpha]$ determined by the edges e_α in $X\backslash T$ with base point $q'(T)$. Applying van Kampen's theorem to determine $\pi_1({}^{X}\!/_{T}\,, q'(T))$ gives a free group with a basis as required. ■

Corollary 1.20. *A maximal tree cannot be contained in any other tree.*

Proof. Let T be a maximal tree in a graph. Suppose T' is another maximal tree that contains T. An edge in $T'\backslash T$ corresponds to a non-trivial element of fundamental group of T'. This contradicts the fact that T' is contractible. ■

This means that the fundamental group of a connected graph as computed in 1.19 is independent of the maximal tree chosen.

Corollary 1.21. *A connected graph is a tree if and only if it is simply connected.*

Proof. Let X be a simply connected graph with a maximal tree T. If X strictly contains T, then X is not simply connected. ■

Corollary 1.22. *A group is free if and only if it is the fundamental group of a graph.*

Proof. If a free group is given on generators $\{a_\alpha\}_{\alpha\in\Lambda}$, then let X be a wedge sum of circles indexed over Λ with base point x_0. Thus X is a graph and $\pi_1(X, x_0)$ is isomorphic to the given free group. ■

We quote the following theorem from [Hatcher, 2002] (p. 85).

Theorem 1.23. *Every covering space of a graph is a graph whose vertices and edges are the lifts of the vertices and edges in the base graph.*

Proof. Let X be a graph and $p : \widetilde{X} \to X$ be a covering map. Denote $X^0 \subset X$ to be the set of vertices of graph X. Take $p^{-1}(X^0)$ to be the set of vertices of $\widetilde{X}$. Consider the continuous map Φ_α associated with the edge e_α defined by the composition $I_\alpha \hookrightarrow X^0 \coprod_{\alpha\in\Lambda} I_\alpha \xrightarrow{q} X$. Each Φ_α is a path in X. By theorem A.28, find a unique lift $\widetilde{\Phi}_\alpha$ of Φ_α passing through each point in $p^{-1}(x)$ for $x \in e_\alpha$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ I_\alpha \arrow[ur, "\widetilde{\Phi}_\alpha"] \arrow[r, "\Phi_\alpha"'] & X \end{tikzcd}$$

The image of a lift $\widetilde{\Phi}_\alpha$ is the closure of a lift of e_α in $\widetilde{X}$. We take the interior of this image to be an edge in $\widetilde{X}$. The two points of $\widetilde{X}$ that this edge joins are lifts of endpoints of $\bar{e}_\alpha$. The graph structure on $\widetilde{X}$ is described by the vertices $p^{-1}(X^0)$ and edges $\mathrm{Int}\,(\mathrm{Im}\,(\widetilde{\Phi}_\alpha))$. Finally it needs to be shown that the the weak topology with respect to these edges is equivalent to the given topology on $\widetilde{X}$. This is evident from the fact that p is a local homeomorphism. ■

1.3 Applications to Free Groups

Results on the fundamental group and covering spaces of a graph lead to geometric realization of a few algebraic properties of a free group.

Theorem 1.24 (Neilsen-Schreier theorem)**.** *Every subgroup of a free group is free.*

Proof. Let F_Λ be a free group with generators $\{a_\alpha\}_{\alpha\in\Lambda}$. Let X be a wedge sum of circles indexed over Λ with base point x_0. Then X is a graph. By theorem 1.19, the fundamental group $\pi_1(X, x_0)$ is isomorphic to F_Λ. By theorem A.35, for each subgroup G of F, we can find a connected covering map $p : (\widetilde{X}_G, \tilde{x}_0) \to (X, x_0)$ such that the image $p_*(\pi_1(\widetilde{X}_G, \widetilde{x}_0))$ of the induced map $p_* : \pi_1(\widetilde{X}_G, \tilde{x}_0) \to \pi_1(X, x_0)$ is isomorphic to G. Theorem A.29 ensures that the map p_* is injective and hence $\pi_1(\widetilde{X}_G, \widetilde{x}_0)$ is isomorphic to G. But theorem 1.23 says that $\widetilde{X}_G$ is a graph. It follows that G is a free group from corollary 1.22. ■

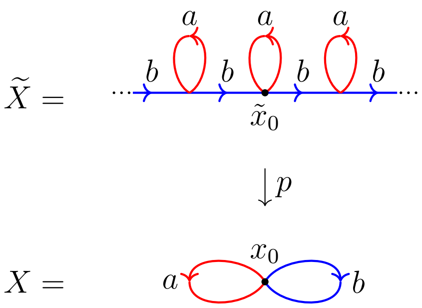


Figure 1.2: The free group on countably many generators is a subgroup of the free group on two generators.

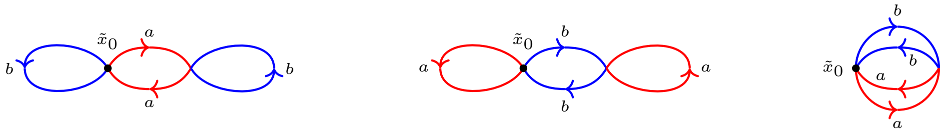


Figure 1.3: All connected double coverings of $S^1 \vee S^1$.

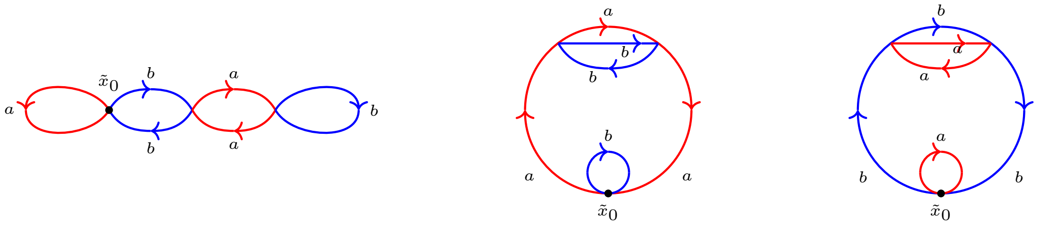


Figure 1.4: All connected non-normal triple coverings of $S^1 \vee S^1$.

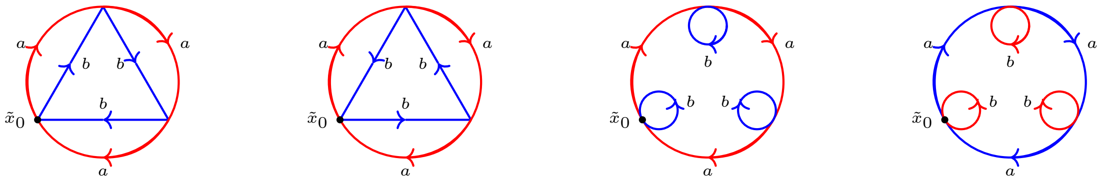


Figure 1.5: All connected normal triple covers of $S^1 \vee S^1$.

Theorem 1.25. *Every free group on countably many generators is a subgroup of free group on two generators.*

Proof. Denote the free group on k generators by F_k for $k \in \mathbb{N}$. Denote the free group on countably many generators $\{g_n\}_{n\in\mathbb{Z}}$ by $F_{\mathbb{Z}}$. Let the wedge of two circles with base point x_0 be denoted as X. Let $\{I_j\}_{j\in\mathbb{Z}}$ and $\{I_k\}_{k\in\mathbb{Z}}$ be two countable collections of unit intervals. Consider the disjoint union $\mathbb{Z} \coprod_j I_j \coprod_k I_k$ and the attaching maps $\phi_j : \partial I_j \to \mathbb{Z}$ and $\phi_k : \partial I_k \to \mathbb{Z}$ defined by the rules $\phi_j(0) =$

j , $\phi_j(1) = (j+1)$ and $\phi_k(0) = k = \phi_k(1)$ for $j, k \in \mathbb{N}$. The resulting quotient space is a graph $\widetilde{X}$ as shown in figure 1.2. Then $p : \widetilde{X} \to X$ is a canonical covering map. Also, from theorem 1.19 it follows that $\pi_1(\widetilde{X}, 0)$ is $F_{\mathbb{Z}}$. Letting the generating classes of loops in $\pi_1(X, x_0)$ to be a and b as shown in the figure 1.2, we therefore have an embedding $\iota : F_{\mathbb{Z}} \hookrightarrow F_2$ defined by $g_n \mapsto b^n a b^{-n}$ for $n \in \mathbb{Z}$. Since F_k is a subgroup of $F_{\mathbb{Z}}$ canonically, it also follows that F_k is a subgroup of F_2. ■

We can enumerate the number of subgroups of the free group on two generators that have a particular finite index, as illustrated by the following proposition.

Proposition 1.26. *Let F_2 be the free group on two generators. Then F_2 contains three subgroups of index 2 and thirteen subgroups of index 3. Of the subgroups of index 3, four are normal subgroups.*

Proof. Let X be the wedge of two circles with base point x_0 as indicated in figure 1.2. We prove the theorem by examining the connected double coverings and connected triple coverings of X. The three connected double coverings of X correspond to the three subgroups of index two in F_2. The seven connected triple coverings of X correspond to seven conjugacy classes of subgroups of index three in F_2. Four of these triple coverings are normal coverings. Taking into consideration the changes in base points, we obtain nine subgroups corresponding to the non-normal connected triple coverings of X. Refer figures 1.3, 1.4 and 1.3. ■

Definition 1.27. *Let X be a graph consisting of finitely many vertices and finitely many edges. The number of vertices minus the number of edges of the graph X is called the* ***Euler characteristic*** *of the graph. It is denoted as $\chi(X)$.*

Denote the rank of a free group F by rank(F). Let T be a maximal tree of a connected graph X with finitely many vertices and finitely many edges. Fix $x_0 \in T$ as the base point. It is easy to see that $\chi(T) = 1$. Also $\chi(X) = \chi(T) - \text{rank}(\pi_1(X, x_0))$ whence $\chi(X) = 1 - \text{rank}(\pi_1(X, x_0))$.

Theorem 1.28. *Let G be a subgroup of the free group F_n on n generators for $n \in \mathbb{N}$. If G has a finite index k in F_n then G is a free group on $1 + k(n-1)$ generators.*

Proof. It was proved that G is a free group. Let X be the wedge of n circles with base point x_0. Then we have a connected covering map $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ such that the fundamental group $\pi_1(\widetilde{X}, \tilde{x}_0)$ is isomorphic to G. Since G has index k in F_n, by theorem A.30, the degree of the covering map p is k. This means that there are k vertices and kn edges in $\widetilde{X}$. Hence $\chi(\widetilde{X}) = 1 - \text{rank}(\pi_1(\widetilde{X}, \tilde{x}_0))$ which gives our result. ■

Corollary 1.29. *The free group on three generators does not contain a free subgroup of finite index on four generators.* ■

1.4 Further Notes and references

The result that every subgroup of a free group is free is attributed to [Nielsen, 1921]. This paper poses the problem combinatorially in terms of non-commuting factors $a_1, \ldots, a_m$, each having an inverse a_i^{-1} and satisfying $a_i a_i^{-1} = a_i^{-1} a_i = 1$. Nielsen's proof, in fact, provides a basis for the subgroup unlike the proof given in this thesis. One can refer to [Stillwell, 1993] (p. 103-104) for Nielsen's proof, where it is outlined as a series of exercises. [Schreier, 1927] proves the same result using another method that also finds the generators of the subgroup of a free group. This method algebraically encodes the process in theorem 1.19 of finding generators of the fundamental group of a graph. Refer [Stillwell, 1993] (p. 105) for further details.

Let n and r be positive integers and let $N(n,r)$ denote the number of subgroups of index n of a free group on r generators. It was shown by [Hall, 1949] that $N(n,r) = n(n!)^{r-1} - \sum_{i=1}^{n-1} [(n-i)!]^{r-1} N(i,r)$. There have been works to prove the same result using graphs and coverings. One can refer to [Nieveen and Smith, 2006] for an accessible proof. The latter paper also proves many other results concerning enumeration of normal subgroups of finite index in a free group, and describes related algorithms.

Chapter 2

CW-complexes

We fix some terminology and notation before proceeding. The n-dimensional closed unit disk of $\mathbb{R}^n$ is denoted by D^n. The interior $\operatorname{int} D^n$ of the n-dimensional unit closed disk is called an n-cell and is also denoted by e^n_α. The 0-dimensional unit closed disk D^0 and the 0-cell e^0 are declared to be the one-point space. The n-dimensional unit sphere S^n is the boundary ∂D^{n+1} of the $(n+1)$-dimensional closed unit disk. The (-1)-dimensional sphere is hence the empty set.

We refer to [Hatcher, 2002] for the definition of a CW-complex.

Definition 2.1. *A topological space X that is constructed in the following way is called a* ***CW-complex****.*

(i) Begin with a discrete set X^0. This set is called the ***0-skeleton*** *of the space X. Each point in X^0 is called a* ***0-cell*** *of X.*

(ii) Form the ***n-skeleton*** *X^n either by taking it to be X^{n-1} or by attaching n-cells e^n_α, for $\alpha \in \Lambda$, to X^{n-1} via a family of maps $\{\phi^n_\alpha : \partial D^n_\alpha \to X^{n-1}\}_\alpha$. In the latter case,*

$$X^n = {X^{n-1} \coprod_\alpha D^n_\alpha} \Big/ {\sim}$$

where $x \sim \phi^n_\alpha(x)$ for $x \in \partial D^n_\alpha$ and $\alpha \in \Lambda$.

(iii) Now let $X = \cup_{n \in \mathbb{N}_0} X^n$. Declare $A \subset X$ to be open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n. This topology is called the weak topology of X with respect to the subspaces X^n.

Declare X^{-1} to be the empty set. Thus the empty set is a CW-complex. This means for $x^0 \in X^0$, the attaching maps $\phi^0_{x^0}$ are the identity maps of the empty set. A CW-complex that has countably many cells is called a **countable**

CW-complex. A CW-complex X is called a **finite-dimensional** CW-complex if $X = X^n$ for some $n \in \mathbb{N}_0$. The smallest such n is called the **dimension** of X. The dimension of the empty set is declared to be (-1). For an n-dimensional CW-complex X, where n is a non-negative integer, we write $\{X^0, \ldots, X^n\}$ as the set of skeleta.

Lemma 2.2. *The quotient and weak topologies agree on a finite-dimensional CW-complex.*

Proof. Let $X = X^n$ where n is the dimension of X. Let $A \subset X$ be in quotient topology of X. Then $q^{-1}(A)$ is open in $X^{n-1} \coprod_\alpha D^n_\alpha$ which gives that $q^{-1}(A)$ is open in X^{n-1} whence $q^{-1}(A) \cap X^{n-1} = A \cap X^{n-1}$ is open. Continue inductively to obtain that A is in the weak topology. If $X = X^n$ has weak topology then $A = A \cap X^n$ is in the quotient topology. Proceed similarly for closed sets. ■

Let $n \in \mathbb{N}_0$ and define q_n to be the quotient map $X^{n-1} \coprod_\alpha D^n_\alpha \to X^n$ sending each point to its equivalence class under the identifications of definition 2.1. The map q_n is not defined if X does not have n-cells.

Definition 2.3. *Let X be a CW-complex. The* ***characteristic map*** *of the cell e^n_α of X is defined to be the composition $\Phi^n_\alpha : D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha \xrightarrow{q_n} X_n \hookrightarrow X$.*

The characteristic maps $\Phi^0_{x^0}$ are obtained to be the inclusion maps $\{x^0\} \hookrightarrow X$. The characteristic map Φ^n_α of the cell e^n_α in X is a continuous map and is an extension of the attaching map ϕ^n_α. Further $\Phi^n_\alpha|_{\text{int}\, D^n_\alpha} : \text{int}\, D^n_\alpha \to e^n_\alpha$ is a homeomorphism. The weak topology on a CW-complex can be equivalently formulated in terms of characteristic maps.

Lemma 2.4. *Let X be a CW-complex. A subset A of the CW complex X is open (or closed) if and only if $(\Phi^n_\alpha)^{-1}(A)$ is open (or closed) in D^n_α for cells e^n_α of X.*

Proof. If A is in the weak topology of X then $(\Phi^n_\alpha)^{-1}(A)$ is open by continuity of Φ^n_α. Now let $A \subset X$ be such that $(\Phi^n_\alpha)^{-1}(A)$ is open in D^n_α for each Φ^n_α. We use induction on n to show that A is open. The base case of $n = 0$ is trivially satisfied. Now suppose that $A \cap X^{n-1}$ is open in X^{n-1}. Consider the quotient map $q_n : X^{n-1} \coprod_\alpha D^n_\alpha \to X^n$. Then $A \cap X^n$ is open in X^n if and only if $q^{-1}(A \cap X^n)$ is open in $X^{n-1} \coprod_\alpha D^n_\alpha$. But this is equivalent to saying $A \cap X^{n-1}$ is open in X^{n-1} and that $(\Phi^n_\alpha)^{-1}(A)$ is open in D^n_α for each α. ■

Corollary 2.5. *The CW-complex X is the quotient space of $\coprod\limits_{n,\alpha} D^n_\alpha$ obtained via the quotient map $\coprod_{n,\alpha} \Phi^n_\alpha$.*

2.1 Examples of CW-complexes

Many topological spaces can be described as CW-complexes by defining the required characteristic maps. However, it needs to be checked whether the existing topology of a space agrees with the weak topology of the given CW-complex structure. This section largely follows [Hatcher, 2002].

Example 2.6. Given any topological space X, a possible CW-complex structure is to take each point in X as a 0-cell. This makes the set X into a discrete space.

Example 2.7 (Graphs)**.** With the tacit understanding that I is homeomorphic to D^1 , graphs are CW-complexes Let us look at a particular graph whose topology is not induced from a euclidean space. Consider the wedge sum of circles $\vee_j S^1$ indexed over $j \in \mathbb{N}$ with the base point x_0. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - 1/j)^2 = j^{-2}, j \in \mathbb{N}\}$. It is easy to see that $\vee_j S^1$ and X can be identified as sets and the canonical identification map $\vee_j S^1 \to X$ is continuous. However, any sequence of points in the interiors of edges of $\vee_j S^1$ is closed. On the other hand, we can have a sequence of non-zero points, with each point from a constituent circle, that converges to the origin in X. Also X is compact but $\vee_j S^1$ is not. To see this, let $q : x_0 \coprod_{j\in\mathbb{N}} I_j \to \vee_j S^1$ to be the quotient map associated with the graph $\vee_j S^1$. Let $A_j = [0, 1/2) \cup (1/2, 1] \subset I_j$ and $B_j = (1/3, 2/3)$. Then $\{q(\cup_j A_j)\} \cup \{q(B_j)\}_j$ for $j \in \mathbb{N}$ is an open cover of $\vee_j S^1$ that does not have a subcover.

It can be shown (see [Hatcher, 2002] p. 86) that the weak topology of a graph with each vertex having only finitely many edges being incident is induced by a euclidean space.

Example 2.8 (n-dimensional sphere and $(n + 1)$-dimensional disk)**.** Let n be a non-negative integer. Consider the n-sphere $S^n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid ||(x_0, x_1, \dots, x_n)||^2 = 1\}$. Let $f : \operatorname{int} D^n \to \mathbb{R}^n$ be the continuous map defined at $x \in \operatorname{int} D^n$ by

$$\begin{cases} x \mapsto \frac{x}{||x||} \tan\left(\frac{\pi}{2}||x||\right) & \text{if } x \neq 0 \text{ and} \\ 0 \mapsto 0. \end{cases}$$

Let σ be the inverse stereographic projection of $\mathbb{R}^n$ onto $S^n \backslash \{(1, 0, \dots, 0)\}$ that is defined at $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ by

$$x = (x_1, \dots, x_n) \mapsto \left(\frac{||x||^2 - 1}{||x||^2 + 1}, \frac{2x_1}{||x||^2 + 1}, \dots, \frac{2x_n}{||x||^2 + 1}\right).$$

Let $\tilde{\sigma} = \sigma \circ f$. Then S^n has a CW-complex structure with the characteristic maps as $\Phi^0 : D^0 \to S^n$ and $\Phi^n : D^n \to S^n$ defined by

$$\Phi^0(*) = (1, 0, \ldots, 0) \text{ and } \Phi^n(x) = \begin{cases} (1, 0, \ldots, 0) \text{ if } x \in \partial D^n \,, \\ \tilde{\sigma}(x) \text{ if } x \in \operatorname{int} D^n. \end{cases}$$

We thus decompose S^n as the disjoint union of cells $e^0 \coprod e^n$. Here the set of skeleta is $\{D^0, \ldots, D^0, S^n\}$. It is easy to see that the CW-complex structure described here agrees with the subspace topology of S^n.

To see that $D^{n+1} \subset \mathbb{R}^{n+1}$ is a CW-complex, the above maps Φ^0 and Φ^n along with the identity map of D^{n+1} are taken to be the required characteristic maps. Consequently, the set of skeleta is $\{D^0, \ldots, D^0, S^n, D^{n+1}\}$. As a set, D^{n+1} is the disjoint union $e^0 \coprod e^n \coprod e^{n+1}$ of cells.

Example 2.9 (n-dimensional sphere)**.** The CW-complex structure on a space X need not be unique. Consider S^n again, with the characteristic maps $\Phi^k_{\pm} : D^k \to S^n$ defined at $x \in D^k$ by

$$\Phi^k_+(x) = (x, \sqrt{1 - ||x||^2}, 0, \ldots, 0) \text{ and}$$
$$\Phi^k_-(x) = (x, -\sqrt{1 - ||x||^2}, 0, \ldots, 0)$$

for $k = 0, \ldots, n$. Under this CW-complex structure, S^n is the disjoint union $e^0_+ \coprod e^0_- \coprod \cdots \coprod e^n_+ \coprod e^n_-$. The set of skeleta is $\{S^0, S^1, \ldots, S^n\}$. Each k-skeleton here contains two k-cells. The weak topology of this CW-complex structure agrees with the subspace topology of S^n.

Example 2.10 (n-dimensional real projective space)**.** Let n be a non-negative integer. The n-dimensional real projective space $\mathbb{R}P^n$ is defined to be the quotient space obtained from S^n via the identifications $v \sim_1 -v$ for $v \in S^n$. It is also defined to be the quotient space obtained from D^n via the identifications $v \sim_2 -v$ for $v \in \partial D^n$.

Since the identifications $\sim_2$ on $\partial D^n = S^{n-1}$ result in $\mathbb{R}P^{n-1}$, we define a CW-complex structure on the quotient space $\mathbb{R}P^n$ with the set of skeleta $\{\mathbb{R}P^0, \ldots, \mathbb{R}P^n\}$ and the characteristic maps as the quotient projection maps $D^k \to \mathbb{R}P^{k-1}$ for $k = 0, \ldots, n$. Thus $\mathbb{R}P^n$ is the disjoint union $e^0 \coprod \ldots \coprod e^n$. Each k-skeleton contains one k-cell.

Example 2.11 (Infinite sphere and infinite-dimensional real projective space)**.** Consider the characteristic maps of example 2.9 for $k \in \mathbb{N}_0$. We thus obtain the space $\cup_{n\in\mathbb{N}_0} S^n$ called as the infinite sphere, denoted by S^∞. The topology on S^∞ is the weak topology with respect to the subspaces S^n. Similarly, considering the characteristic maps of example 2.10 for $k \in \mathbb{N}_0$, we obtain the infinite-dimensional real projective space $\mathbb{R}P^\infty = \cup_{n\in\mathbb{N}_0}\mathbb{R}P^n$. It is easy to see that $\mathbb{R}P^\infty$ is the quotient space of S^∞. The infinite sphere occurs in other topological contexts too. In section 2.3 , we will compare the various topologies on the infinite sphere.

2.2 Products of CW-complexes

This section is sourced from [Lundell and Weingram, 2012](p. 26-27 and p. 56-57). Let X and Y be two CW-complexes. Denote the p-cells of X by e^p_α for $\alpha \in \Lambda$. Similarly denote the q-cells of Y by f^q_β for $\beta \in \Omega$. Denote the respective characteristic maps of X as Φ^p_α and the respective characteristic maps of Y as Ψ^q_β. We will build a CW-complex $X \times_{CW} Y$ from X and Y as follows. For this, note that $D^n \cong D^p \times D^q$ and $\partial D^n \cong (\partial D^p \times D^q) \cup (D^p \times \partial D^q)$ for all non-negative integers p, q and n such that $p + q = n$.

(i) Let the product space $X^0 \times Y^0$ of 0-skeleta of X and Y be the 0-skeleton $(X \times_{CW} Y)^0$.

(ii) Construct the n-skeleton $(X \times_{CW} Y)^n$ from $(X \times_{CW} Y)^{n-1}$ by attaching the cells $e^p_\alpha \times f^q_\beta$ such that $p + q = n$ via the restriction of characteristic maps

$$\Phi^p_\alpha \times \Psi^q_\beta : (\partial D^p_\alpha \times D^q_\beta) \cup (D^p_\alpha \times \partial D^q_\beta) \to (X \times_{CW} Y)^{n-1}.$$

In such a case, as a set

$$(X \times_{CW} Y)^n = (X \times_{CW} Y)^{n-1} \coprod_{\substack{\alpha,\beta,\\ p+q=n}} e^p_\alpha \times f^q_\beta.$$

If no cells e^p_α and f^q_β exist such that $p + q = n$, then let $(X \times_{CW} Y)^n = (X \times_{CW} Y)^{n-1}$.

(iii) Set $X \times_{CW} Y = \cup_{n\in\mathbb{N}_0} (X \times_{CW} Y)^n$ with the weak topology with respect to the subspaces $(X \times_{CW} Y)^n$.

Lemma 2.12. *Let X and Y be CW-complexes. The identity map $X \times_{CW} Y \to X \times Y$ is a continuous map*

Proof. We note that as sets $(X \times_{CW} Y) = X \times Y$. Further, the projection maps $X \times_{CW} Y \to X$ and $X \times_{CW} Y \to Y$ are continuous. The result follows. ■

In general, the weak topology on $X \times_{CW} Y$ has more open sets than the product topology. The following example from [Dowker, 1952](p. 563-564) illustrates this.

Example 2.13. Let X be a graph with uncountably many edges incident on a vertex x_0 and Y be a graph with countably many edges incident on a vertex y_0. Further let the closures of these edges be homeomorphic to I (refer lemma 1.6). Index the closures A_i of edges incident on x_0 by sequences $i = (i_1, i_2, \ldots)$ of integers. Index the closures B_j of edges incident on y_0 by $j \in \mathbb{N}$. Parametrize A_i by I_i and B_j by I_j using the corresponding characteristic maps such that x_0 is the image of $0 \in I_i$ and y_0 is the image of $0 \in I_j$. Consider the collection $P = \{(1/i_j, 1/i_j) \in A_i \times B_j\}_{(i,j)}$ of points in $X \times Y$. Since the intersection of P with $A_i \times B_j$ for each pair (i, j) is a point, the set P is closed in $X \times_{CW} Y$. However P is not closed in product space $X \times Y$. We claim that (x_0, y_0) is in the closure of P in $X \times Y$. For this, we will show that any neighborhood of (x_0, y_0) in the product space $X \times Y$ contains a point in P. Let $U \times V$ be a basic product open neighborhood of (x_0, y_0) in $X \times Y$. A basic open neighborhood U of x_0 in X is the union of open neighborhoods $[0, a_i)$ of $0 \in I_i$ for $a_i \in (0, 1)$. A basic open neighborhood V of y_0 in Y is the union of open neighborhoods $[0, b_j)$ of $0 \in I_j$ for $b_j \in (0, 1)$. Let the index $\hat{\imath} = (\hat{\imath}_1, \hat{\imath}_2, \ldots)$ be chosen such that $\hat{\imath}_j \geq \max\{j, 1/b_j\}$ for each j. Choose the index $\hat{\jmath}$ such that $\hat{\jmath} \geq 1/a_{\hat{\imath}}$. Then $U \times V$ contains $(1/\hat{\imath}_{\hat{\jmath}}, 1/\hat{\imath}_{\hat{\jmath}}) \in P$ because $1/\hat{\imath}_{\hat{\jmath}}$ belongs to both $[0, a_{\hat{\imath}})$ and $[0, b_{\hat{\jmath}})$.

Both weak topology and product topology on $X \times_{CW} Y$ agree in, among others, one case. We have the following from [Milnor, 1956a](p. 272).

Lemma 2.14. *Product of countable CW-complexes is a CW-complex.*

Example 2.15. Let $X = S^m \times S^n$. Let $p_0 = (1, 0, \ldots, 0) \in S^m$ and $q_0 = (1, 0, \ldots, 0) \in S^m$. Consider the CW-complex structures of example 2.8 on S^m and S^n. Denote the characteristic maps of S^m as Φ^0 and Φ^m. Denote the characteristic maps of S^n as Ψ^0 and Ψ^n. A CW-complex structure on X given by the characteristic maps $\Phi^0 \times \Psi^0$, $\Phi^0 \times \Psi^n$, $\Phi^m \times \Psi^0$ and $\Phi^m \times \Psi^n$; this agrees with the subspace topology of X from $\mathbb{R}^{m+1} \times \mathbb{R}^{n+1}$. One could also begin with the CW-complex structure of example 2.9 on S^m and S^n.

2.3 The infinite sphere

In this section, we will show that the infinite sphere S^∞ in example 2.11 can be obtained as a subspace of countable product of $\mathbb{R}$. Also, we will look at the group S^1 acting on S^∞.

Let $\mathbb{R}^\omega$ denote the countable product of $\mathbb{R}$. The standard bounded metric on $\mathbb{R}$ is defined as $\bar{d}(a,b) = \min\{|a-b|, 1\}$ for $a, b \in \mathbb{R}$. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in $\mathbb{R}^\omega$. The product topology on $\mathbb{R}^\omega$ is induced by the product metric

$$D(x,y) = \sup_{i\in\mathbb{N}} \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}.$$

A basic open set in product topology is of the form $\prod_i U_i$, where each U_i is an open subset of $\mathbb{R}$ and only finitely many of U_i are proper subsets of $\mathbb{R}$. The uniform topology on $\mathbb{R}^\omega$ is induced by the metric

$$\rho(x,y) = \sup_{i\in\mathbb{N}} \{\bar{d}(x_i, y_i)\}.$$

Denote $\ell^2(\mathbb{R})$ to be the subset consisting of all sequences $x = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ such that $\sum_{i\in\mathbb{N}} x_i^2$ converges. The topology induced by the norm

$$||x||_{\ell^2} = \left[\sum_{i\in\mathbb{N}} x_i^2 \right]^{\frac{1}{2}}$$

on $\ell^2(\mathbb{R})$ is called the ℓ^2-topology. Apart from this topology, $\ell^2(\mathbb{R})$ also inherits product topology and uniform topology from $\mathbb{R}^\omega$. It is a well-known fact that $\ell^2(\mathbb{R})$ in ℓ^2-topology is a Hilbert space ([Kreyszig, 1989] p. 133). The inner product that induces the ℓ^2-norm is defined by

$$\langle x, y \rangle = \sum_{i\in\mathbb{N}} x_i y_i$$

for $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in \ell^2(\mathbb{R})$. Also known ([Munkres, 2000] p. 127-128) is that these three topologies follow the inclusions

$$\text{product topology} \subset \text{uniform topology} \subset \ell^2\text{-topology}.$$

Let S denote the unit sphere $\{x \in \ell^2(\mathbb{R}) \mid ||x||_{\ell^2} = 1\}$ in $\ell^2(\mathbb{R})$.

To compare the various topologies inherited by S from $\ell^2(\mathbb{R})$, we will make use of the following two lemmas regarding convergences in $\ell^2(\mathbb{R})$. The former lemma is from [Kreyszig, 1989](p. 261) and the latter is from [Bessaga and Peł czyński, 1975](p. 47).

Lemma 2.16. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* $\ell^2(\mathbb{R})$ *and let* $x \in \ell^2(\mathbb{R})$*. Then* $(\langle x_n, y\rangle)_n$ *converges to* $\langle x, y\rangle$ *for* $y \in \ell^2(\mathbb{R})$ *if and only if*

(i) the sequence $(||x_n||_{\ell^2})_n$ *is bounded, and*

(ii) $(x_n)_n$ *converges to* x *coordinate-wise.*

Lemma 2.17. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* $\ell^2(\mathbb{R})$ *and* $x \in \ell^2(\mathbb{R})$ *such that* $\langle x_n, y\rangle$ *converges to* $\langle x, y\rangle$ *for* $y \in \ell^2(\mathbb{R})$*, and* $||x_n||_{\ell^2}$ *converges to* $||x||_{\ell^2}$*. Then* $||x_n - x||_{\ell^2}$ *converges to zero.*

The next lemma is from [Bessaga and Peł czyński, 1975](p. 174).

Lemma 2.18. *The product, uniform and* ℓ^2*-topologies on* S *inherited from* $\ell^2(\mathbb{R})$ *are equivalent.*

Proof. On $\ell^2(\mathbb{R})$ we have the inclusions, product topology $\subset$ uniform topology $\subset$ ℓ^2-topology. Hence it suffices to show that coordinate-wise convergence of sequences in S implies convergence in ℓ^2-norm. Choosing $(x_n)_{n\in\mathbb{N}}$ to be a sequence in S that converges to $x \in S$ coordinate-wise, the above two lemmas give the required result. ■

Theorem 2.19. *The subspace*

$$\{x \in S \mid x = (x_1, x_2, \dots, x_i, \dots) \text{ such that } x_i \text{ vanishes for all but finitely many } i\}$$

of unit sphere S *in* $\ell^2(\mathbb{R})$ *is the infinite sphere* S^∞*.*

Proof. As all topologies on S inherited from $\ell^2(\mathbb{R})$ are equivalent, let us consider S with product topology. Then it is easy to see that on S^∞ the weak topology with respect to the subspaces S^n agrees with the subspace topology inherited from S. ■

Corollary 2.20. *The infinite-dimensional real projective space* $\mathbb{R}P^\infty$ *is the quotient space obtained from the infinite sphere* S^∞ *via the identifications* $x \sim -x$ *for* $x \in S^\infty$*.* ■

Finite-dimensional spheres are not contractible; they have non-trivial homotopy groups. The situation, however, is different for S and S^∞. The following result is from [Hatcher, 2002](p. 88).

Theorem 2.21. *The unit sphere of $\ell^2(\mathbb{R})$ and the infinite sphere S^∞ are contractible.*

Proof. Consider the continuous linear operator $T : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ defined by $(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$ for $x = (x_1, x_2, \ldots) \in \ell^2(\mathbb{R})$. Define $F : \ell^2(\mathbb{R}) \times I \to \ell^2(\mathbb{R})$ by $F(x,t) = (1-t)x + tT(x)$ for $(x,t) \in \ell^2(\mathbb{R}) \times I$. Define $G : \ell^2(\mathbb{R}) \times I \to \ell^2(\mathbb{R})$ by $G(x,t) = (1-t)T(x) + t(1, 0, \ldots)$ for $(x,t) \in \ell^2(\mathbb{R}) \times I$. The required homotopy $H : S \times I \to S$ is defined by

$$H(x,t) = \begin{cases} \dfrac{F(x,2t)}{||F(x,2t)||}, & \text{if } 0 \leq t \leq \frac{1}{2} \\ \\ \dfrac{G(x,2t-1)}{||G(x,2t-1)||}, & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

To see that S^∞ is contractible, replace S by S^∞ in the above homotopy. ■

We end this chapter with defining a special kind of map between CW-complexes and give an example of such a map. The example also occurs in more important contexts; it will be referred to in chapter 5.

Definition 2.22. *A continuous map $f : X \to Y$ of CW-complexes is said to be cellular if it carries the k-skeleton of X into the k-skeleton of Y, that is, $f(X^n) \subset Y^n$ for $n \in \mathbb{N}_0$.*

Lemma 2.23. *There is a canonical free left-action of S^1 on $\ell^2(\mathbb{R})$ that is continuous with respect to the product, uniform and ℓ^2-topologies. Further it preserves S and S^∞. The group action $S^1 \times S^\infty \to S^\infty$ is a cellular map.*

Proof. Define $f : S^1 \times \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ by

$$\begin{aligned} &(e^{i\theta}, (x_1, y_1, x_2, y_2, \ldots)) \\ &\quad \mapsto (\text{Re } e^{i\theta}(x_1 + iy_1), \text{Im } e^{i\theta}(x_1 + iy_1), \text{Re } e^{i\theta}(x_2 + iy_2), \text{Im } e^{i\theta}(x_2 + iy_2), \ldots) \end{aligned}$$

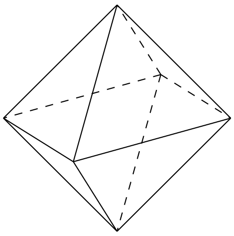
for $e^{i\theta} \in S^1$ and $(x_1, y_1, x_2, y_2, \ldots) \in \ell^2(\mathbb{R})$.

Let d generically denote the product metric, uniform metric or the metric induced by ℓ^2-norm on $\ell^2(\mathbb{R})$. The product topology on $S^1 \times \ell^2(\mathbb{R})$ is given by the metric $\tilde{d}\left((e^{i\theta}, z), (e^{i\alpha}, w)\right) = |e^{i\theta} - e^{i\alpha}| + d(z, w)$ for $(e^{i\theta}, z), (e^{i\alpha}, w) \in S^1 \times \ell^2(\mathbb{R})$. Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}(1 + d(w, 0))^{-1}$ so that $d\left(e^{i\theta} \cdot z, e^{i\alpha} \cdot w\right) < \epsilon$ whenever $\tilde{d}\left((e^{i\theta}, z), (e^{i\alpha}, w)\right) < \delta$. It is easy to check that this choice of δ works by noting that the metric d on $\ell^2(\mathbb{R})$ satisfies the properties $d\left(e^{i\theta} \cdot z, e^{i\theta} \cdot w\right) = d(z, w)$ and $d(e^{i\theta} \cdot z, e^{i\alpha} \cdot z) \leq |e^{i\theta} - e^{i\alpha}| d(z, 0)$.

The second part of the theorem is true for both CW complex structures of examples 2.8 and 2.9. ■

Chapter 3

Joins



An octahedron can be regarded as the space of line segments joining e thpoints of the equatorial square to the apical points such that two line segments intersect, if at all, only at the end points. This space of line segments is called a join. Thus, a circle can be thought of (up to homeomorphism) as a join of the set of unit vectors of x-axis and the set of unit vectors of y-axis, on a plane.

For realizing join in a more general setting, it is most natural to think about this space of line segments in a vector space. Hence consider two non-empty subsets X_1 and X_2 of a topological vector space V with subspace topology. Motivated with the examples of an octahedron and a circle, we construct join of spaces X_1 and X_2 by taking union of line segments joining points in X_1 to points in X_2 such that if two line segments meet, then they meet only at the end points.

However, this construction is raw and unwieldy. If we want to construct join of two intersecting subsets of a vector space, the condition on the intersection of a pair of line segments is impossible to satisfy. Even for disjoint subsets this strange condition seems elusive, like in the case of join of two compact intervals of the real line. Moreover, it is not clear that this construction is independent of the ambient vector space. Perhaps if the construction were independent, we can look for join of two compact intervals of real line by embedding them in a higher dimensional vector space. Nonetheless, with these issues resolved, the notion of join seems to be associated only with vector spaces.

In this chapter, we will construct join of arbitrary topological spaces in various ways and compare the topologies of these constructions. Only non-empty topological spaces are considered in this chapter. Exposition here is mainly based on [Brown, 2006] with [Hatcher, 2002], [Milnor, 1956b] and [Fritsch and Golasiński, 2004] as other references.

3.1 Join of two spaces

Definition 3.1. *Let X_1 and X_2 be topological spaces that can be embedded in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. The* ***join*** $\boldsymbol{J(X_1, X_2)}$ *of spaces X_1 and X_2 is defined to be the subspace of line segments in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}$ joining points in $X_1 \times \{0\} \times \{0\}$ to points in $\{0\} \times X_2 \times \{1\}$. That is, the join $J(X_1, X_2)$ is given by*

$$\{(tx_1, (1-t)x_2, (1-t)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R} \mid x_1 \in X_1, x_2 \in X_2, t \in [0,1]\}$$

with the subspace topology inherited from $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}$.

The above definition resolves the issue of existence of join for a pair of subsets of a vector space; one moves to a higher dimensional space to construct their join. Examining this construction closely, we see that any point p of join $J(X_1, X_2)$ lies on some line segment, say, joining $(x_1, 0, 0) \in X_1 \times \{0\} \times \{0\}$ and $(0, x_2, 1) \in \{0\} \times X_2 \times \{1\}$. Thus p can be seen as the triad (x_1, x_2, t) where t determines the position of point p on this line segment. If t is neither zero nor one, this triad is unique. But if t is zero, there is no unique choice of x_1. This is because a point p of $J(X_1, X_2)$ has its t parameter zero if and only if p lies in $\{0\} \times X_2 \times \{1\} \subset J(X_1, X_2)$. Similarly, if a point of $J(X_1, X_2)$ has t parameter one, there is no unique choice of x_2 to describe the point as a triad.

We have the following from [Hatcher, 2002](p. 9).

Definition 3.2. *Let X_1 and X_2 be topological spaces. The* ***join*** $\boldsymbol{X_1 * X_2}$ *of spaces X_1 and X_2 is defined to be the quotient space obtained from the product space $X_1 \times X_2 \times I$ via the identifications $(x_1, x_2, 0) \sim (x_1', x_2, 0)$ and $(x_1, x_2, 1) \sim (x_1, x_2', 1)$ for $x_1, x_1' \in X_1$ and $x_2, x_2' \in X_2$.*

We have the following from [Milnor, 1956b](p. 430).

Definition 3.3. *Let X_1 and X_2 be topological spaces. The* ***join*** $\boldsymbol{X_1 \circ X_2}$ *of X_1 and X_2 is defined as the collection of points described as formal convex combinations $tx_1 \oplus (1-t)x_2$ for $x_1 \in X_1, x_2 \in X_2$ and $t \in [0,1]$. If $t = 0$, then x_1 is chosen arbitrarily or omitted. If $t = 1$, then x_2 is chosen arbitrarily or omitted. The* *topology on join $X_1 \circ X_2$ is the smallest topology such that the coordinate maps*

$$\theta : X_1 \circ X_2 \to I \quad , \quad \chi_1 : \theta^{-1}((0,1]) \to X_1 \quad \textit{and} \quad \chi_2 : \theta^{-1}([0,1)) \to X_2$$

are continuous.

A subbasis for the topology on join $X_1 \circ X_2$ is given by the union of the following kinds of sets.

(i) $\theta^{-1}(U) = \{tx_1 \oplus (1-t)x_2 \mid x_1 \in X_1, x_2 \in X_2, t \in U\}$ for U open in $[0,1]$.

(ii) $\chi_1^{-1}(U) = \{tx_1 \oplus (1-t)x_2 \mid x_1 \in U, x_2 \in X_2, t \in (0,1]\}$ for U open in X_1.

(iii) $\chi_2^{-1}(U) = \{tx_1 \oplus (1-t)x_2 \mid x_1 \in U, x_2 \in X_2, t \in [0,1)\}$ for U open in X_2.

Let f be a function into the join $X_1 \circ X_2$. Call the maps $\theta \circ f$, $\chi_1 \circ f$ and $\chi_2 \circ f$, defined on appropriate domains, as the **coordinates** of f. Thus f is continuous if and only if the coordinates of f are continuous.

Let X_1 and X_2 be Hausdorff spaces. Are the joins $X_1 * X_2$ and $X_1 \circ X_2$ Hausdorff? The answer is affirmative for $X_1 \circ X_2$. The author does not know the answer in case of $X_1 * X_2$.

The following is from [Brown, 2006](p. 171).

Lemma 3.4. *Let X_1 and X_2 be Hausdorff topological spaces. Then the join $X_1 \circ X_2$ is Hausdorff.*

Proof. Let $x = tx_1 \oplus (1-t)x_2$ and $y = sy_1 \oplus (1-s)y_2$ be two distinct points of $X_1 \circ X_2$ where $x_1, y_1 \in X_1, x_2, y_2 \in X_2$, and $t, s \in [0,1]$. If $t \neq s$, find open sets U_t and U_s in I that separate t and s respectively. Then $\theta^{-1}(U_t)$ and $\theta^{-1}(U_s)$ are open sets in $X_1 \circ X_2$ that separate x and y respectively. If $t = s \neq 0$, find open sets U_{x_1} and U_{y_1} in X_1 that separate x_1 and y_1 respectively. Then $\chi_1^{-1}(U_{x_1})$ and $\chi_1^{-1}(U_{y_1})$ are open sets in $X_1 \circ X_2$ that separate x and y respectively. If $t = s = 0$, find open sets U_{x_2} and U_{y_2} in X_2 separating x_2 and y_2 respectively. The sets $\chi_2^{-1}(U_{x_2})$ and $\chi_2^{-1}(U_{y_2})$ are open sets in $X_1 \circ X_2$ that separate x and y respectively in $X_1 \circ X_2$. ■

Now we will show that various joins constructed are equivalent as sets. We will further compare their topologies.

Lemma 3.5. *Let X_1 and X_2 be subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Then there exists a canonical bijection from the join $X_1 * X_2$ onto the join $J(X_1, X_2)$ that is continuous. If X_1 and X_2 are compact, then this map is a homeomorphism.*

Proof. Let q be the quotient map from $X_1 \times X_2 \times I$ onto $X_1 * X_2$ sending each point (x_1, x_2, t) to its equivalence class $[(x_1, x_2, t)]$. Define g to be the map from $X_1 \times X_2 \times I$ to $J(X_1, X_2)$ that sends (x_1, x_2, t) to the point $(tx_1, (1-t)x_2, 1-t)$. The map g is well-defined, surjective and continuous (consider sequences). On the collection of points in $X_1 \times X_2 \times I$ with t neither zero nor one, g is injective. Furthermore, for every x_1 and x_2, the map q collapses each fiber $g^{-1}(x_1, 0, 0)$ and $g^{-1}(0, x_2, 1)$. Hence from theorem A.4, there exists a well-defined continuous bijection h defined by $[(x_1, x_2, t)] \mapsto (tx_1, (1-t)x_2, 1-t)$ such that the following diagram commutes.

$$\begin{tikzcd}
X_1 \times X_2 \times I \arrow[r, "g"] \arrow[d, "q"'] & J(X_1, X_2) \\
X_1 * X_2 \arrow[ur, "h"'] &
\end{tikzcd}$$

If X_1 and X_2 are compact then so is $X_1 * X_2$. The space $J(X_1, X_2)$ is Hausdorff. Thus follows the second part of the theorem. ■

Lemma 3.6. *Let X_1 and X_2 be two topological spaces. Then there exists a canonical bijection from the join $X_1 * X_2$ onto the join $X_1 \circ X_2$ that is continuous. If X_1 and X_2 are compact and Hausdorff, then this map is a homeomorphism.*

Proof. Define $g : X_1 \times X_2 \times I \to X_1 \circ X_2$ by $(x_1, x_2, t) \mapsto tx_1 \oplus (1-t)x_2$. Let $q : X_1 \times X_2 \times I \to X_1 * X_2$ be the quotient map sending (x_1, x_2, t) to its equivalence class $[(x_1, x_2, t)]$. The map g induces a well-defined bijection $h : X_1 * X_2 \to X_1 \circ X_2$ such that the following diagram commutes.

$$\begin{tikzcd}
X_1 \times X_2 \times I \arrow[r, "g"] \arrow[d, "q"'] & X_1 \circ X_2 \\
X_1 * X_2 \arrow[ur, "h"'] &
\end{tikzcd}$$

The map h is continuous if and only if g is continuous. Consider the coordinates of g. We have

$$\theta \circ g : (x_1, x_2, t) \mapsto t,$$
$$\chi_1 \circ g : (x_1, x_2, t) \mapsto x_1, \text{ and}$$
$$\chi_2 \circ g : (x_1, x_2, t) \mapsto x_2.$$

The map $\theta \circ g$ is defined on $X_1 \times X_2 \times I$. The map $\chi_1 \circ g$ is defined at points (x_1, x_2, t) with t non-zero, and the map $\chi_2 \circ g$ is defined at the points (x_1, x_2, t)

with $(1-t)$ non-zero. We see that the domains of $\chi_1 \circ g$ and $\chi_2 \circ g$ are open sets in $X_1 \times X_2 \times I$. Certainly the coordinates of g are continuous and hence the map g is continuous. If X_1 and X_2 are compact spaces then $X_1 * X_2$ is compact. If X_1 and X_2 are Hausdorff then $X_1 \circ X_2$ is Hausdorff. Thus follows the second part of the theorem. ■

Lemma 3.7. *Let X_1 and X_2 be subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Then there exists a canonical bijection from the join $J(X_1, X_2)$ onto the join $X_1 \circ X_2$ that is continuous. If X_1 and X_2 are compact, then this map is a homeomorphism.*

Proof. Define the map $h : J(X_1, X_2) \to X_1 \circ X_2$ by

$$(tx_1, (1-t)x_2, 1-t) \mapsto tx_1 \oplus (1-t)x_2.$$

Considering the coordinates of the map h, it is easy to see that h is a continuous map. The second part of the theorem follows from the previous two lemmas. ■

Example 3.8. Let X_1 and X_2 be two copies of the unit closed interval. To construct the join $X_1 * X_2$, consider the cube $X_1 \times X_2 \times I$, as shown in figure 3.1. We collapse the face $X_1 \times X_2 \times \{0\}$ onto $\{0\} \times X_2 \times \{0\}$, and $X_1 \times X_2 \times \{1\}$ onto $X_1 \times \{0\} \times \{1\}$. The joins $J(X_1, X_2)$, $X_1 * X_2$ and $X_1 \circ X_2$ are homeomorphic, and hence $X_1 * X_2$ is the tetrahedron in $\mathbb{R}^3$ as shown.

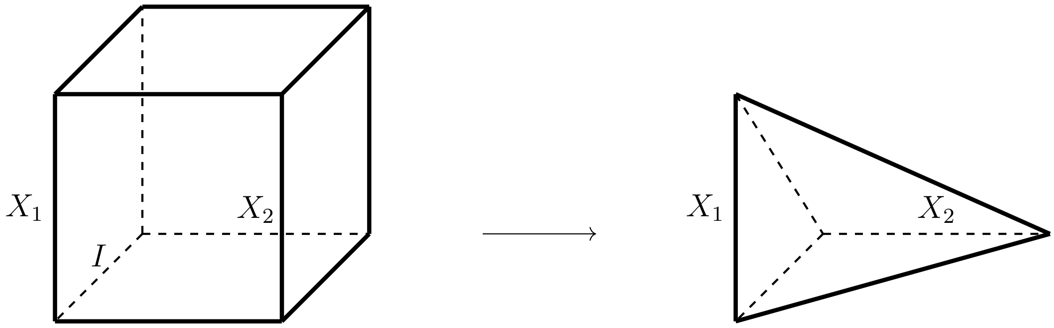
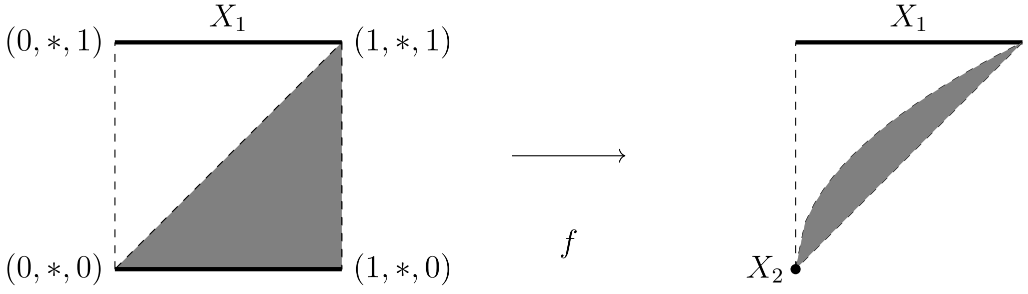


Figure 3.1: Join of closed interval with itself.

Now we consider examples that show that the inclusions among various topologies on join are strict.

Example 3.9. Let $X_1 = (0, 1)$ and $X_2 = \{*\}$. Refer figure 3.2. The join $J(X_1, X_2)$ is an open triangular region along with a side and its opposite vertex included. The side and the opposite vertex are X_1 and X_2, respectively, considered as subspaces of $J(X_1, X_2)$. To construct $X_1 * X_2$, we begin with $X_1 \times X_2 \times I$, which is a square region with a pair of opposite sides included and all vertices deleted. Consider the quotient map $q : X_1 \times X_2 \times I \to X_1 * X_2$ and the map $g : X_1 \times X_2 \times I \to J(X_1, X_2)$ as in the proof of theorem 3.5. Let U be the gray open region in $X_1 \times X_2 \times I$ as shown. The image $q(U)$ is open in $X_1 * X_2$. However, the image $f(U)$ is not open in $J(X_1, X_2)$ because it contains the vertex X_2; any open set of $J(X_1, X_2)$ containing the vertex X_2 must contain the entire angle described about it.



$(0, 1) \times \{*\} \times I$

$J(X_1, X_2)$

Figure 3.2: Joins $X * Y$ and $J(X, Y)$ are not homeomorphic.

Example 3.10. Let $X_1 = \mathbb{Q}$ and $X_2 = \{*\}$. Consider the joins $X_1 \circ X_2$ and $J(X_1, X_2)$. Let $\epsilon \in (0, 1)$. Then $\theta^{-1}([0, \epsilon)) = \{tx \oplus (1-t)* \mid 0 \leq t < \epsilon, x \in \mathbb{Q}\} = \cup_{x \in \mathbb{Q}} \{tx \oplus (1-t)* \mid t < \frac{\epsilon}{x}\}$ is not open in $J(X_1, X_2)$.

3.2 Join of multiple spaces

The joins $J(X_1, X_2)$ and $X_1 \circ X_2$ can be seamlessly generalized for multiple spaces. However, there seems to be no clear way of generalizing $X_1 * X_2$. We will see how other notions of joins, when generalized, offer a consistent way of defining join of multiple spaces as a quotient space.

Definition 3.11. *Let $X_1, \ldots, X_n$ be topological spaces such that there exist inclusion maps $X_j \hookrightarrow \mathbb{R}^{k_j}$ for $j = 1, \ldots, n$. Embed each X_j in $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n} \times \mathbb{R}^{n-1}$ by mapping $x \in X_j$ to the point*

$$(0, \ldots, 0, x_j, 0, \ldots, 0, e_{j-1})$$

that has x_j at the j^{th} position, and e_{j-1} is the unit vector in $\mathbb{R}^{n-1}$ that has 1 at the $(j-1)^{th}$ position. Define the ***n-fold join*** *$J(X_1, \ldots, X_n)$ to be the set of points*

$t_1x_1 + \cdots + t_nx_n$ for $x_j \in X_j$ and non-negative real numbers $t_1, \ldots, t_n$ such that $t_1 + \cdots + t_n = 1$. The n-fold join is given the subspace topology inherited from $\mathbb{R}^{k_1} \times \ldots \times \mathbb{R}^{k_n} \times \mathbb{R}^{n-1}$.

To generalize above definition to an arbitrary family of spaces, we will use functional notation for describing points in a Cartesian product ([Munkres, 2000] p. 113). Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be an indexed family of spaces. The Cartesian product $\prod_{\alpha\in\Lambda} A^\alpha$ is regarded as the set of all functions

$$f : \Lambda \to \bigcup_{\alpha\in\Lambda} A_\alpha$$

such that $f(\alpha) \in A_\alpha$ for each α. If all the spaces A_α are equal to one set, say A, then the Cartesian product $\prod_{\alpha\in\Lambda} A_\alpha$ of this family is the set A^Λ of all Λ-tuples $f : \Lambda \to A$.

Definition 3.12. *Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be an indexed family of topological spaces such that there exist inclusion maps $X_\alpha \hookrightarrow \mathbb{R}^{n_\alpha}$ for each α. Consider the product space $\prod_{\alpha\in\Lambda} \mathbb{R}^{n_\alpha} \times \mathbb{R}^\Lambda$. For each α, let $\iota_\alpha : X_\alpha \hookrightarrow \prod_{\alpha\in\Lambda} \mathbb{R}^{n_\alpha} \times \mathbb{R}^\Lambda$ be the embedding that maps $x_\alpha \in X_\alpha$ to the point $\iota_\alpha(x_\alpha) = (\iota^1_\alpha(x_\alpha), \iota^2_\alpha(x_\alpha))$ where*

$$\iota^1_\alpha(x_\alpha) : \Lambda \to \bigcup_{\alpha\in\Lambda} \mathbb{R}^{n_\alpha} \textit{ is defined by } \begin{cases} \iota^1_\alpha(x_\alpha)(\alpha) = x_\alpha, \\ \iota^1_\alpha(x_\alpha)(\beta) = 0 \in \mathbb{R}^{n_\beta} \textit{ such that } \beta \neq \alpha \end{cases},$$

$$\textit{and } \iota^2_\alpha(x_\alpha) : \Lambda \to \mathbb{R} \textit{ is defined by } \begin{cases} \iota^2_\alpha(x_\alpha)(\alpha) = 1, \\ \iota^2_\alpha(x_\alpha)(\beta) = 0 \textit{ for } \beta \in \Lambda \textit{ such that } \beta \neq \end{cases}.$$

*Define the Λ-**fold join** $J(X_\alpha)_{\alpha\in\Lambda}$ to be the set of points $\sum_{\alpha\in\Lambda} t_\alpha \iota_\alpha(x_\alpha)$. Each point has all but finitely many non-negative real parameters t_α vanishing and satisfies $\sum_{\alpha\in\Lambda} t_\alpha = 1$. If a parameter t_α of a point $\sum_\alpha t_\alpha \iota_\alpha(x_\alpha)$ is zero, then $x_\alpha \in X_\alpha$ is chosen arbitrarily or omitted. The join $J(X_\alpha)_{\alpha\in\Lambda}$ is given the subspace topology inherited from $\prod_{\alpha\in\Lambda} \mathbb{R}^{n_\alpha} \times \mathbb{R}^\Lambda$.*

Without any loss of clarity, we will write a point of $J(X_\alpha)_\alpha$ as $\sum_\alpha t_\alpha x_\alpha$. When the indexing set Λ is finite, the above definition is not equivalent to the earlier definition of n-fold join. The points of n-fold join of definition 3.11 have their last coordinate in $\mathbb{R}^{n-1}$ whereas the points of $\{1, \ldots, n\}$-fold join of definition 3.12 have their last coordinate in $\mathbb{R}^n$. However, both constructions are canonically equivalent due to the constraint that the non-negative reals $t_1, \ldots, t_n$ add up to one.

Definition 3.13. *Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be an indexed family of topological spaces. Define the join $\circ_{\alpha\in\Lambda}X_\alpha$ to be the set of points each of which is described by*

(i) non-negative real parameters t_α that vanish for all but finitely many α and satisfy $\sum_{\alpha\in\Lambda} t_\alpha = 1$; and

(ii) values $x_\alpha \in X_\alpha$ for α such that t_α is non-zero.

Each point of $\circ_{\alpha\in\Lambda}X_\alpha$ is denoted as $\oplus_{\alpha\in\Lambda}t_\alpha x_\alpha$. If a parameter t_α of a point $\oplus_{\alpha\in\Lambda}t_\alpha x_\alpha$ is zero, then $x_\alpha \in X_\alpha$ is chosen arbitrarily or omitted in this notation.
The topology on $\circ_{\alpha\in\Lambda}X_\alpha$ is the smallest one such that the coordinate functions

$$\theta_\alpha : \circ_{\alpha\in\Lambda}X_\alpha \to [0,1] \quad \textit{and} \quad \chi_\alpha : \theta_\alpha^{-1}((0,1]) \to X_\alpha$$

are continuous for every α.

Let f be a function into the join $\circ_{\alpha\in\Lambda}X_\alpha$. Call the maps $\theta_\alpha \circ f$ and $\chi_\alpha \circ f$, defined on appropriate domains, for $\alpha \in \Lambda$ as the **coordinates** of f. Thus f is continuous if and only if the coordinates of f are continuous.

Are join operations, when indexing set Λ is finite, of definitions 3.11 and 3.13 associative? The answer is yes, up to homeomorphism.

We have the following theorem from [Brown, 2006](p. 170).

Theorem 3.14. *Let $X_1, \dots, X_n$ be topological spaces. Then there exists a canonical homeomorphism*

$$h_i : (X_1 \circ \dots \circ X_i) \circ (X_{i+1} \circ \dots \circ X_n) \to X_1 \circ \dots \circ X_n$$

for $i = 1, \dots, n$.

Proof. For $i \in \{1, \dots, n\}$, let h_i be the mapping

$$\begin{aligned} x = r(s_1x_1 \oplus \dots \oplus s_ix_i) \oplus (1-r)(s_{i+1}x_{i+1} \oplus \dots \oplus s_nx_n) \\ \mapsto rs_1x_1 \oplus \dots \oplus rs_ix_i \oplus (1-r)s_{i+1}x_{i+1} \oplus \dots \oplus (1-r)s_nx_n \end{aligned}$$

where $r, s_j \in [0,1]$, $x_j \in X_j$ for $j = 1, \dots, n$ and $s_1 + \dots + s_i = 1 = s_{i+1} + \dots + s_n$. The map h_i is well-defined and bijective. Let us check that the coordinates of h_i are continuous. Consider the coordinate maps, for $j \leq i$,

$$\theta_j : x \mapsto \begin{cases} rs_j\,, & \text{if } r \neq 0\,, \\ 0\,, & \text{if } r = 0\,, \end{cases} \quad \text{and} \quad \chi_j : x \mapsto x_j.$$

Here χ_j is defined only at the points x with $r \neq 0$ and $s_j \neq 0$. Hence χ_j is the composition $x \mapsto s_1x_1 \oplus \cdots \oplus s_ix_i \mapsto x_j$ of continuous maps. For continuity of θ_j, first consider the case $r \neq 0$. The collection of all points x with $r \neq 0$ forms an open set. On this set, θ_j is the product of the continuous maps $x \mapsto r$ and $x \mapsto s_1x_1 \oplus \ldots \oplus s_ix_i \mapsto s_j$. Now let $r = 0$. For $\delta \in (0,1]$, the set $\theta_j^{-1}([0,\delta))$ is given by the union $\{x \mid 0 \leq s_j \leq 1\} \cup_{r\in(0,1]} \{x \mid 0 < s_j < \delta/r\}$ of open sets. This completes the case for $j \leq i$. Now consider the coordinate maps, for $j > i$,

$$\theta_j : x \mapsto \begin{cases} (1-r)s_j & \text{if } (1-r) \neq 0 \\ 0 & \text{if } (1-r) = 0 \end{cases} \quad \text{and} \quad \chi_j : x \mapsto x_j.$$

Here χ_j is defined at the points x with $(1-r) \neq 0$ and $s_j \neq 0$. Continuity of θ_j and χ_j for $j > i$ follows from arguments similar to the case $j \leq i$.
Now to show that h_i is a homeomorphism, define the inverse map by

$$x = r_1x_1 \oplus \cdots \oplus r_nx_n \mapsto r(s_1x_1 \oplus \cdots \oplus s_ix_i) \oplus (1-r)(s_{i+1}x_{i+1} \oplus \cdots \oplus s_nx_n)$$

where $r_j \in [0,1]$, $x_j \in X_j$ for $j = 1,\ldots,n$ such that $r_1 + \cdots + r_n = 1$ and $r := r_1 + \cdots + r_i$,

$$s_j := \begin{cases} r^{-1}r_j\,, & \text{if } r \neq 0\,, \\ 0\,, & \text{if } r = 0\,, \end{cases} \text{for } 1 \leq j \leq i,$$

$$s_j := \begin{cases} (1-r)^{-1}r_j\,, & \text{if } (1-r) \neq 0\,, \\ 0\,, & \text{if } (1-r) = 0\,, \end{cases} \text{for } i+1 \leq j \leq n.$$

Consider the coordinates of the inverse map. The map $x \mapsto r$ is the sum of the continuous maps $x \mapsto r_j$ for $1 \leq j \leq i$. The map $x \mapsto s_1x_1 \oplus \cdots \oplus s_ix_i$ is defined at the points x with $r \neq 0$, whence this map can be written as the product of the maps $x \mapsto r_1x_1 \oplus \cdots \oplus r_ix_i$ and $x \mapsto r^{-1}$. Similarly, the coordinate $x \mapsto s_{i+1}x_{i+1} \oplus \cdots \oplus s_nx_n$ is continuous. ■

Corollary 3.15. *Let X_1, X_2 and X_3 be topological spaces. Then the joins $(X_1 \circ X_2) \circ X_3$ and $X_1 \circ (X_2 \circ X_3)$ are homeomorphic.* ■

Theorem 3.16. *Let $X_1, \ldots, X_n$ be topological spaces. Then there exists a canonical homeomorphism*

$$h_i : J(J(X_1,\ldots,X_i), J(X_{i+1},\ldots,X_n)) \to J(X_1,\ldots,X_n)$$

for $i = 1, \dots, n$.

Proof. For $i \in \{1, \dots, n\}$, define h_i by the rule

$$\begin{aligned}(r(s_1x_1, \dots, s_ix_i, s_2, \dots, s_i), (1-r)(s_{i+1}x_{i+1}, \dots, s_nx_n, s_{i+2}, \dots, s_n), 1-r) \\ \mapsto (rs_1x_1, \dots, rs_ix_i, (1-r)s_{i+1}x_{i+1}, \dots, (1-r)s_nx_n, \\ rs_2, \dots, rs_i, (1-r)s_{i+1}, \dots, (1-r)s_n)\end{aligned}$$

where $r, s_j \in [0,1]$, $x_j \in X_j$ for $j = 1, \dots, n$ and $s_1 + \cdots + s_i = 1 = s_{i+1} + \cdots + s_n$. It follows from sequential arguments that h_i is continuous. Now define the inverse map of h_i as

$$\begin{aligned}(r_1x_1, \dots, r_nx_n, r_2, \dots, r_n) \mapsto \\ (r(s_1x_1, \dots, s_ix_i, s_2, \dots, s_i), (1-r)(s_{i+1}x_{i+1}, \\ \dots, s_nx_n, s_{i+2}, \dots, s_n), 1-r)\end{aligned}$$

where $r_j \in [0,1]$, $x_j \in X_j$ for $j = 1, \dots, n$ such that $r_1 + \cdots + r_n = 1$ and $r := r_1 + \cdots + r_i$,

$$s_j := \begin{cases} r^{-1}r_j\,, & \text{if } r \neq 0\,, \\ 0\,, & \text{if } r = 0\,, \end{cases} \quad \text{for } 1 \leq j \leq i,$$

$$s_j := \begin{cases} (1-r)^{-1}r_j\,, & \text{if } (1-r) \neq 0\,, \\ 0\,, & \text{if } (1-r) = 0\,, \end{cases} \quad \text{for } i+1 \leq j \leq n.$$

It again follows from sequential arguments that the inverse map of h_i is continuous. ■

Corollary 3.17. *Let* X_1, X_2 *and* X_3 *be topological spaces that can be embedded in euclidean spaces. Then the joins* $J(J(X_1, X_2), X_3)$ *and* $J(X_1, J(X_2, X_3))$ *are homeomorphic.* ■

Theorem 3.18. *Let* $\{X_j\}_{j\in\mathbb{N}}$ *be a countably infinite family of topological spaces. Then there exists a canonical homeomorphism*

$$h_i : (\underset{j\leq i}{\circ} X_j) \circ (\underset{j>i}{\circ} X_j) \to \underset{j\geq 1}{\circ} X_j$$

for $i \in \mathbb{N}$.

Proof. For $i \in \mathbb{N}$, let h_i be the mapping defined by

$$x = r(\bigoplus_{j\leq i} s_j x_j) \oplus (1-r)(\bigoplus_{j>i} x_k) \mapsto \bigoplus_{j\geq 1} r s_j x_j \oplus_{k>i} (1-r) s_j x_j$$

where $r, s_j \in [0,1], x_j \in X_j$ for $j \in \mathbb{N}$, all but finitely many s_j vanish, and $\sum_{j\leq i} s_j = 1 = \sum_{j>i} s_j$. The map h_i is well-defined and bijective. Let us check that the coordinates of h_i are continuous. The coordinate maps, for $j \leq i$, are

$$\theta_j : x \mapsto \begin{cases} r s_j\,, & \text{if } r \neq 0\,, \\ 0\,, & \text{if } r = 0\,, \end{cases} \quad \text{and} \quad \chi_j : x \mapsto x_j.$$

The coordinate maps, for $j > i$ are

$$\theta_j : x \mapsto \begin{cases} (1-r) s_j & \text{if } (1-r) \neq 0 \\ 0 & \text{if } (1-r) = 0 \end{cases} \quad \text{and} \quad \chi_j : x \mapsto x_j.$$

The continuity of these coordinate maps is proved as in theorem 3.14. Now to show that h_i is a homeomorphism, define the inverse map by

$$x = \bigoplus_{j\geq 1} r_j x_j \mapsto r(\bigoplus_{j\leq i} s_i x_i) \oplus (1-r)(\bigoplus_{j<i} s_j x_j)$$

where $r_j \in [0,1]$, $x_j \in X_j$ for $j \in \mathbb{N}$, all but finitely many r_j vanish, $\sum_{j\geq 1} r_j = 1$ and $r \coloneqq r_1 + \cdots + r_i$,

$$s_j \coloneqq \begin{cases} r^{-1} r_j\,, & \text{if } r \neq 0\,, \\ 0\,, & \text{if } r = 0\,, \end{cases} \quad \text{for } 1 \leq j \leq i,$$

$$s_j \coloneqq \begin{cases} (1-r)^{-1} r_j\,, & \text{if } (1-r) \neq 0\,, \\ 0\,, & \text{if } (1-r) = 0\,, \end{cases} \quad \text{for } j > i.$$

The continuity of coordinates of the inverse map too is proved in theorem 3.14. ■

Theorem 3.19. *Let* $\{X_j\}_{j\in\mathbb{N}}$ *be a countably infinite family of topological spaces. Then there exists a canonical homeomorphism*

$$h_i : J(J(X_j)_{j\leq i}\,,\, J(X_j)_{j>i}) \to J(X_j)_{j\geq 1}$$

for $i \in \mathbb{N}$.

Proof. For $i \in \mathbb{N}$, define h_i by the rule

$$\left(r\left(\sum_{j\le i} s_j x_j \right), (1-r)\left(\sum_{j>i} s_j x_j \right), 1-r \right) \mapsto \sum_{j\le i} r s_j x_j + \sum_{j>i} (1-r) s_j x_j$$

where $r, s_j \in [0,1]$, $x_j \in X_j$ for $j \in \mathbb{N}$, all but finitely many s_j, for $j > i$ vanish, and $\sum_{j\le i} s_j = 1 = \sum_{j>i} s_j$. It follows from sequential arguments that h_i is continuous. Now define the inverse map of h_i as

$$\sum_{j\ge 1} r_j x_j \mapsto \left(r\left(\sum_{j\le i} s_j x_j \right), (1-r)\left(\sum_{j>i} s_j x_j \right), 1-r \right)$$

where $r_j \in [0,1]$, $x_j \in X_j$ for $j \in \mathbb{N}$, all but finitely many r_j vanish, $\sum_{j\ge 1} r_j = 1$ and $r := r_1 + \cdots + r_i$,

$$s_j := \begin{cases} r^{-1} r_j\,, & \text{if } r \neq 0\,, \\ 0\,, & \text{if } r = 0\,, \end{cases} \quad \text{for } 1 \le j \le i,$$

$$s_j := \begin{cases} (1-r)^{-1} r_j\,, & \text{if } (1-r) \neq 0\,, \\ 0\,, & \text{if } (1-r) = 0\,, \end{cases} \quad \text{for } j > i.$$

It again follows from sequential arguments that the inverse map of h_i is continuous. ■

We have an analogue of lemma 3.4 for multiple spaces.

Theorem 3.20. *Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a family of Hausdorff spaces. Then the join $\circ_{\alpha\in\Lambda} X_\alpha$ is a Hausdorff space.*

Proof. Let x and y be two distinct points of $\circ_\alpha X_\alpha$. For some natural numbers n and m, we write $x = \oplus_{i=1}^{n} t_{\alpha_{ix}} x_{\alpha_{ix}}$ and $y = \oplus_{j=1}^{m} s_{\alpha_{jy}} y_{\alpha_{jy}}$ where $\alpha_{ix}, \alpha_{jy} \in \Lambda$, $x_{\alpha_{ix}} \in X_{\alpha_{ix}}$, $y_{\alpha_{jy}} \in X_{\alpha_{jy}}$, $t_{\alpha_{ix}}, t_{\alpha_{jy}} \in (0,1]$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ such that $\sum_{i=1}^{n} t_{\alpha_{ix}} = \sum_{j=1}^{m} s_{\alpha_{jy}} = 1$.
Case 1 If $\alpha_{ix} \neq \alpha_{jy}$ for some pair i, j then the open sets $\theta_{\alpha_{ix}}^{-1}((0,1))$ and $\theta_{\alpha_{jy}}^{-1}((0,1))$ in $\circ_\alpha X_\alpha$ separate x and y respectively.
Case 2 Let $m = n$ and $\alpha_{ix} = \alpha_{iy} := \alpha_i$ for $i = 1, \ldots, n$. If $t_{\alpha_\ell} \neq s_{\alpha_\ell}$ for some $\ell \in \{1, \ldots, n\}$ then separate these two parameters respectively by open sets U_t and U_s in $(0,1]$. The open sets $\theta_{\alpha_\ell}^{-1}(U_t)$ and $\theta_{\alpha_\ell}^{-1}(U_s)$ in $\circ_\alpha X_\alpha$ separate x and y respectively. If $t_{\alpha_i} = s_{\alpha_i}$ for $i = 1, \ldots, n$ then $x_{\alpha_k} \neq y_{\alpha_k}$ for some $k \in \{1, \ldots, n\}$. Let V_x and V_y be open sets in X_{α_k} that separate x_{α_k} and y_{α_k} respectively. Then the open sets $\chi_{\alpha_k}^{-1}(V_x)$ and $\chi_{\alpha_k}^{-1}(V_y)$ in $\circ_\alpha X_\alpha$ separate x and y respectively. ■

Now let us generalize the definition 3.2 for multiple spaces.

Definition 3.21. *For $j = 1, \ldots, n+1$, let X_j be the singleton containing the unit vector e_j of $\mathbb{R}^{n+1}$ that has 1 as its j^{th} coordinate. Define the n-simplex Δ^n to be the join $J(X_1, \ldots, X_{n+1})$. That is, Δ^n is given by*

$$\left\{ (t_1, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{j=1}^{n+1} t_j = 1, \text{ and } 0 \leq t_j \leq 1 \text{ for each } j \right\}.$$

We see that Δ^1 is homeomorphic, via the rule $(t, 1-t) \mapsto t$, to the unit closed interval. Thus the join $X_1 * X_2$ can be regarded as the quotient space obtained by identifying points of $X_1 \times X_2 \times \Delta^1$. We also note that given a point $(t_1 x_1, \ldots, t_n x_n, t_2, \ldots, t_n)$ of join $J(X_1, \ldots, X_n)$, the vector $(t_1, \ldots, t_n)$ belongs to Δ^{n-1}.

Definition 3.22. *Let $X_1, \ldots, X_n$ be topological spaces. Define the join $X_1 * \cdots * X_n$ to be the quotient space obtained from product space $X_1 \times \cdots \times X_n \times \Delta^{n-1}$ via the identifications $(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n, e_j) \sim (x'_1, \ldots, x'_{j-1}, x_j, x'_{j+1}, \ldots, x'_n, e_j)$ for $x_j, x'_j \in X_j$ and $j = 1, \ldots, n$.*

Loosely speaking, we consider a copy of $X_1 \times \cdots \times X_n$ at each point of Δ^{n-1} and for $j = 1, \ldots, n$, collapse the copy placed at $e_j \in \Delta^{n-1}$ onto $X_1 \times \cdots X_{j-1} \times X_{j+1} \times \cdots X_n$.

Definition 3.23. *Let Λ be an indexing set and consider the product space $\mathbb{R}^\Lambda$. Let X_α be the singleton containing the point*

$$e_\alpha : \Lambda \to \mathbb{R} \text{ defined by } \begin{cases} e_\alpha(\alpha) = 1 \text{ and} \\ e_\alpha(\beta) = 0 \text{ for } \beta \in \Lambda \text{ such that } \beta \neq \alpha. \end{cases}$$

Define the Λ-simplex Δ^Λ to be the join $J(X_\alpha)_{\alpha\in\Lambda}$. That is, Δ^Λ is given by

$$\left\{ t : \Lambda \to \mathbb{R} \;\middle|\; 0 \leq t \leq 1, \text{ all but finitely many } t(\alpha) \text{ vanish and } \sum_{\alpha\in\Lambda} t(\alpha) = 1 \right\}$$

with the subspace topology inherited from $\mathbb{R}^\Lambda$.

Definition 3.24. *Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be an indexed family of topological spaces. Define the join $*_{\alpha\in\Lambda} X_\alpha$ to be the quotient space obtained from the product space $\prod_{\alpha\in\Lambda} X_\alpha \times \Delta^\Lambda$ via the identifications, for each $\alpha \in \Lambda$, $(f_\alpha, e_\alpha) \sim (f'_\alpha, e_\alpha)$ for $f_\alpha, f'_\alpha \in \prod_\alpha X_\alpha$ such that $f_\alpha(\alpha) = f'_\alpha(\alpha)$.*

Finally, we compare these topologies. We have the following results.

Theorem 3.25. *Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be an indexed family of topological spaces such that there exist inclusion maps $X_\alpha \hookrightarrow \mathbb{R}^{n_\alpha}$ for each α. Then there exists a canonical bijection from the join $*_{\alpha\in\Lambda}X_\alpha$ onto the join $J(X_\alpha)_{\alpha\in\Lambda}$ that is continuous. If the spaces X_α are compact, then this map is a homeomorphism.*

Proof. Let q be the quotient map from $\prod_{\alpha\in\Lambda} X_\alpha \times \Delta^\Lambda$ onto $*_{\alpha\in\Lambda}X_\alpha$ sending each point (f,t) to $[(f,t)]$. Define g from $\prod_{\alpha\in\Lambda} X_\alpha \times \Delta^\Lambda$ to $J(X_\alpha)_{\alpha\in\Lambda}$ that sends (f,t) to the point $\sum_\alpha t(\alpha)f(\alpha)$. The continuous map g induces a well-defined continuous bijection $h : *_{\alpha\in\Lambda}X_\alpha \to J(X_\alpha)_{\alpha\in\Lambda}$ such that the following diagram commutes.

$$\begin{tikzcd}
\prod_\alpha X_\alpha \times \Delta^\Lambda \arrow[r, "g"] \arrow[d, "q"'] & J(X_\alpha)_\alpha \\
*_\alpha X_\alpha \arrow[ur, "h"'] &
\end{tikzcd}$$

The Λ-simplex Δ^Λ is a closed subspace of $[0,1]^\Lambda$. If $\{X_\alpha\}_\alpha$ is a collection of compact spaces then $\prod_{\alpha\in\Lambda} X_\alpha \times \Delta^\Lambda$ is compact. Thus follows the second part of the theorem. ∎

Theorem 3.26. *Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be an indexed family of topological spaces. Then there exists a canonical bijection from the join $*_{\alpha\in\Lambda}X_\alpha$ onto the join $\circ_{\alpha\in\Lambda}X_\alpha$ that is continuous. If the spaces X_α are compact, then this map is a homeomorphism.*

Proof. Consider the continuous map $g : \prod_{\alpha\in\Lambda} X_\alpha \times \Delta^\Lambda \to \circ_{\alpha\in\Lambda}X_\alpha$ defined by $(f,t) \mapsto \oplus_\alpha t(\alpha)f(\alpha)$. Let q be the quotient map from $\prod_{\alpha\in\Lambda} X_\alpha \times \Delta^\Lambda$ onto $*_{\alpha\in\Lambda}X_\alpha$ sending each point (f,t) to $[(f,t)]$. The map g induces a well-defined continuous bijection $h : *_{\alpha\in\Lambda}X_\alpha \to \circ_{\alpha\in\Lambda}X_\alpha$ such that the following diagram commutes.

$$\begin{tikzcd}
\prod_\alpha X_\alpha \times \Delta^\Lambda \arrow[r, "g"] \arrow[d, "q"'] & \circ_\alpha X_\alpha \\
*_\alpha X_\alpha \arrow[ur, "h"'] &
\end{tikzcd}$$

If $\{X_\alpha\}$ is a collection of compact spaces then $*_{\alpha\in\Lambda}X_\alpha$ is compact. If $\{X_\alpha\}$ is a collection of Hausdorff spaces then $\circ_{\alpha\in\Lambda}X_\alpha$ is Hausdorff. Thus follows the second part of the theorem. ∎

Lemma 3.27. *Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be an indexed family of topological spaces such that there exist inclusion maps $X_\alpha \hookrightarrow \mathbb{R}^{n_\alpha}$ for each α. Then there exists a canonical* *bijection from the join $J(X_\alpha)_{\alpha\in\Lambda}$ onto the join $\circ_{\alpha\in\Lambda}X_\alpha$ that is continuous. If the spaces X_α are compact, then this map is a homeomorphism.*

Proof. Define the map $h : J(X_\alpha)_{\alpha\in\Lambda} \to \circ_{\alpha\in\Lambda}X_\alpha$ by

$$\sum_\alpha t_\alpha x_\alpha \mapsto \oplus_\alpha t_\alpha x_\alpha.$$

Considering the coordinates of the map h, it is easy to see that h is a continuous map. The second part of the theorem follows from the previous two lemmas. ■

We do have the following simple case when the various topologies of joins of spaces agree.

Theorem 3.28. *Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a family of discrete topological spaces. Then the joins $*_\alpha X_\alpha$ and $\circ_\alpha X_\alpha$ are homeomorphic. If each X_α can be embedded in a euclidean space, then these joins are homeomorphic to the join $J(X_\alpha)_\alpha$.*

Proof. It suffices to prove that the canonical identity map $h : *_\alpha X_\alpha \to \circ_\alpha X_\alpha$ of theorem 3.24 is an open map. Consider the quotient map q from $\prod_{\alpha\in\Lambda} X_\alpha \times \Delta^\Lambda$ onto $*_{\alpha\in\Lambda}X_\alpha$ sending each point (f,t) to $[(f,t)]$. Let $U \subset \Delta^\Lambda$ be an open set. Then $q(\{f\} \times U)$ is mapped to

$$\left(\bigcup_{\alpha\in\Lambda} \chi_\alpha^{-1}\left(f(\alpha)\right)\right) \bigcap \left(\bigcup_{\substack{\alpha\in\Lambda \\ t\in U}} \theta_\alpha^{-1}\left(t(\alpha)\right)\right)$$

which is an open set in $\circ_\alpha X_\alpha$. Since any open set in $*_\alpha X_\alpha$ can be written as the union of sets of the form $q(\{f\} \times U)$ for $f \in \prod_{\alpha\in\Lambda} X_\alpha \times \Delta^\Lambda$ and U open in Δ^Λ, this finishes the proof. ■

3.3 Homotopy groups of joins

Lemma 3.29. *Let X_1 and X_2 be two topological spaces. Then the joins $X_1 * X_2$ and $X_1 \circ X_2$ are path connected. If X_1 and X_2 can be embedded in a euclidean space, then $J(X_1, X_2)$ is path connected.*

Proof. It suffices to prove that $X_1 * X_2$ is path connected as there exist canonical continuous identity maps from $X_1 * X_2$ onto the other joins. Fix two points $[(a, *, 1)]$ and $[(*, b, 0)]$ in $X_1 * X_2$ where $*$ denotes an arbitrary choice of coordinate. Let $[(x_1, x_2, t)] \in X_1 * X_2$ be given such that $t \neq 0$. Then the path

$\gamma : I \to X_1 * X_2$ defined by

$$s \mapsto \begin{cases} [(x_1, x_2, (1-2s)t + 2s)] & \text{for } s \in [0, \frac{1}{2}], \\ [(x_1, b, 2-2s)] & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

joins $[(x_1, x_2, t)]$ to $[(*, b, 0)]$. Let $[(y_1, y_2, t)] \in X_1 * X_2$ be given such that $t \neq 1$. Then the path $\delta : I \to X_1 * X_2$ defined by

$$s \mapsto \begin{cases} [(y_1, y_2, (1-2s)t)] & \text{for } s \in [0, \frac{1}{2}], \\ [(a, y_2, 2s-1)] & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

joins $[(y_1, y_2, t)]$ to $[(a, *, 1)]$. The points $[(*, b, 0)]$ and $[(a, *, 1)]$ can be joined by the path $s \mapsto [(a, b, s)]$. ■

Lemma 3.30. *Let X_1 and X_2 be path connected topological spaces. Then the joins $X_1 * X_2$ and $X_1 \circ X_2$ are simply connected. If X_1 and X_2 can be embedded in a euclidean space, then $J(X_1, X_2)$ is simply connected.*

Proof. We will use van Kampen's theorem to prove that $X_1 * X_2$ is simply connected. Let $q : X_1 \times X_2 \times I \to X_1 * X_2$ be the quotient map sending each point to its equivalence class. There exist canonical inclusions $X_1 \hookrightarrow X_1 * X_2$ and $X_2 \hookrightarrow X_1 * X_2$. Let the base point be $u = (u_1, u_2, \frac{1}{2})$ for some $u_1 \in X_1$ and $u_2 \in X_2$. Let $A = q(X_1 \times X_2 \times (0, 1])$ and $B = q(X_1 \times X_2 \times [0, 1))$. The sets A and B are open, path connected and cover $X_1 * X_2$. Also, $A \cap B = q(X_1 \times X_2 \times (0, 1))$ is path connected and contains the base point. The set A deformation retracts onto X_1 via the homotopy $A \times I \to A$ defined by $([(x_1, x_2, t)], s) \mapsto [(x_1, x_2, (1-s)t + s)]$. The set B deformation retracts to X_2 via the homotopy $B \times I \to B$ defined by $([(x_1, x_2, t)], s) \mapsto [(x_1, x_2, (1-s)t)]$. The set $A \cap B$ deformation retracts to the set $q(X_1 \times X_2 \times \{\frac{1}{2}\})$ via the homotopy $A \cap B \times I \to A \cap B$ defined by

$$([(x_1, x_2, t)], s) \mapsto [(x_1, x_2, (1-s)t + \tfrac{s}{2})].$$

Hence $\pi_1(A, u) * \pi_1(B, u) = \pi_1(X_1, u_1) * \pi_1(X_2, u_2)$ and $\pi_1(A \cap B, u) = \pi_1(X_1 \times X_2, (u_1, u_2)) = \pi_1(X_1, u_1) \times \pi_1(X_2, u_2)$. The inclusion maps $\iota_1 : A \cap B \hookrightarrow A$ and $\iota_2 : A \cap B \hookrightarrow B$ induce projection maps $\iota_{1*} : \pi_1(X_1, u_1) \times \pi_1(X_2, u_2) \to \pi_1(X_1, u_1)$ and $\iota_{2*} : \pi_1(X_1, u_1) \times \pi_1(X_2, u_2) \to \pi_1(X_2, u_2)$ respectively. The normal subgroup generated by the elements $\gamma_1(\gamma_2)^{-1}$, for $\gamma_1 \times \gamma_2 \in \pi_1(A \cap B, u)$, in $\pi_1(X_1, u_1) * \pi_1(X_2, u_2)$ is the whole group. By van Kampen's, $\pi_1(X_1 * X_2, u)$ is trivial.

In a similar vein, it can be proved that $X_1 \circ X_2$ and $J(X_1, X_2)$ are simply connected.

■

In fact, we have a stronger result that we quote from Milnor[Milnor, 1956b]. The proof is skipped.

Theorem 3.31. *Let* $X_1, \ldots, X_{n+1}$ *be topological spaces such that each space* X_i *is* $(c_i - 1)$ *connected. Then the joins* $X_1 * \cdots * X_{n+1}$ *and* $X_1 \circ \cdots \circ X_{n+1}$ *are* $(\sum_{i=1}^{n+1} c_i + n - 1)$*-connected. The corresponding result holds true for* $J(X_1, \ldots, X_{n+1})$ *if each* X_i *can be embedded in a euclidean space. In particular, join of* $(n+1)$ *spaces is at least* $(n-1)$*-connected.*

Finally, we have the following whose proof is an easy consequence of corollaries 3.18 and 3.19.

Theorem 3.32. *Let* $\{X_j\}_{j \in \mathbb{N}}$ *be a countably infinite family of topological spaces. Then the join* $\circ_j X_j$ *is* ∞*-connected. If each of the spaces* X_j *can be embedded in a euclidean space, then an analogous result holds true for* $J(X_j)_j$.

Proof. Choose $n \in \mathbb{N}$ arbitrarily. Since $\circ_j X_j$ is homeomorphic to the join $X_1 \circ \cdots X_n \circ (\underset{j>n\circ_j}{X})$ of $(n+1)$ spaces, it is at least $(n-1)$ connected. ■

3.4 Further notes and references

We see that all constructions of join of multiple spaces can be canonically identified as sets. However, in general, the topologies differ, as seen in the examples of section 3.1. Moreover, associativity does not hold true for the join defined as a quotient space. Quoting [Hatcher,] , “This is another instance of how mixing product and quotient constructions can lead to bad point-set topological behavior”.

The “technical awkwardness” of not possessing associativity is rectified by working in another class of spaces, called k-spaces, with a redefined notion of product of spaces. Refer [Brown, 2006] (section 5.9) and [Fritsch and Golasiński, 2004] (p. 471) for more details. The latter source also compares the various topological joins (p. 469-470). In fact, it shows that all the constructions of joins of two spaces are homotopy equivalent (p. 470).

The author of this thesis could not prove the join of locally compact spaces with quotient topology is associative, as cited in [Fritsch and Golasiński, 2004]. Neither could the author construct examples showing that the join of arbitrary spaces with quotient topology does not satisfy associativity.

Chapter 4

Fiber Bundles

This chapter is a superficial introduction to the notions of fiber bundles and principal G-bundles. The exposition is limited to what will be required for construction of classifying space of a group in chapter 5. The references are [Husemoller, 1994] and [Hatcher, 2002].

4.1 Fiber bundles

Definition 4.1. *A continuous map $p : E \to B$ of topological spaces is said to be a **fiber bundle** with fiber F if every point $b \in B$ has an open neighborhood U and a homeomorphism $\Phi_U : p^{-1}(U) \to U \times F$ such that $pr_U \circ \Phi_U = p$, that is, the following diagram commutes.*

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\Phi_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr_U"] \\
& U &
\end{tikzcd}$$

The space E is called the **total space** and B is called the **base space**. Since F is homeomorphic to $p^{-1}(b)$ for each $b \in B$, the fiber might not be mentioned in contexts where it is clear from the map p.

The maps Φ_U indicate that the total space E locally looks like the product space $B \times F$. Indeed, the projection map $\text{pr}_B : B \times F \to B$ is a fiber bundle. This projection map is called the **trivial fiber bundle** over space B with fiber F. Hence the maps Φ_U are called **local trivializations** of the fiber bundle p.

Example 4.2. Consider a covering map $p : E \to B$. If B is not connected, fibers over each point in B might not be homeomorphic to each other. If B is connected then p is a fiber bundle. For $b \in B$, choose U to be the evenly covered neighborhood of b with respect to p; the corresponding local trivialization is $p : p^{-1}(U) \to U \times p^{-1}(b)$ is defined canonically.

Example 4.3. Consider the infinite Möbius strip M, that is, the quotient space obtained from $I \times \mathbb{R}$ with the identifications $(0, t) \sim (1, -t)$. Let C be the subspace $\{[(s, 0)] \mid s \in I\}$ of M and $p : M \to C$ be the canonical projection map. Then p is a continuous surjection with fiber $\mathbb{R}$. The local trivializations are given by

$$\Phi_1 : p^{-1}(U_1) \to U_1 \times \mathbb{R} \text{ where}$$
$$U_1 = \{[(s,0)] \mid s \in [0, 1/2) \cup (1/2, 1]\} \ ,$$
$$[(s,t)] \mapsto ([(s,0)], t) \ \text{ for } s \in [0, 1/2) \text{ and } [(s,t)] \mapsto ([(s,0)], -t) \ \text{ for } s \in (1/2, 1], \text{ and}$$

$$\Phi_2 : p^{-1}(U_2) \to U_2 \times I \text{ where}$$
$$U_2 = \{[(s,0)] \mid s \in (0,1)\} \ \text{and}$$
$$[(s,t)] \mapsto ([(s,0)], t)$$

Let us examine the behavior of local trivializations closely. Let p be a fiber bundle with fiber F. Suppose we have two local trivializations Φ_i and Φ_j with domains U_i and U_j respectively. Further let the domains U_i and U_j intersect non-trivially. Restricting the trivializations to $U_i \cap U_j$, we obtain the following commutative diagram.

$$\begin{tikzcd}
(U_i \cap U_j) \times F \arrow[dr, "\mathrm{pr}_i"'] & p^{-1}(U_i \cap U_j) \arrow[l, "\Phi_i"'] \arrow[r, "\Phi_j"] \arrow[d, "p"'] & (U_i \cap U_j) \times F \\
& U_i \cap U_j \arrow[ur, "\mathrm{pr}_j"'] &
\end{tikzcd}$$

Therefore the map $\Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ is a homeomorphism. It is called a **transition map** and is denoted by Φ_{ji}. In case of example 4.3, transition map Φ_{21} can be made explicit. We obtain $\Phi_{21} : (U_1 \cap U_2) \times \mathbb{R} \to (U_1 \cap U_2) \times \mathbb{R}$ with $([(s,0)], t)$ mapping to itself whenever $0 < s < 1/2$, and to $([(s,0)], -t)$ whenever $1/2 < s < 1$. In this example, we observe that for each point in $U_1 \cap U_2$, the transition map Φ_{21} reparametrizes $\mathbb{R}$. Generalizing this observation, let $\tilde{x} \in p^{-1}(x)$ be mapped to (x_i, f_i) by Φ_i and to (x_j, f_j) by Φ_j. These trivializations commute with projection maps; therefore $x_i = x_j = x$. Thus the transition map Φ_{ji} is identity over the first coordinate. Therefore, for each point x in $U_i \cap U_j$, the transition map is a homeomorphism of the fiber as seen in the following commutative diagram.

$$\begin{tikzcd}
\{x\} \times F \arrow[dr, "\mathrm{pr}_i"'] & \tilde{x} \arrow[l, "\Phi_i"'] \arrow[r, "\Phi_j"] \arrow[d, "p"] & \{x\} \times F \\
& x \arrow[ur, "\mathrm{pr}_j"'] &
\end{tikzcd}$$

Transition maps are obtained for every pair of trivializations Φ_i and Φ_j. For each transition map Φ_{ji}, we have the associated map

$$\begin{aligned} \tilde{\Phi}_{ji} : (U_i \cap U_j) \times F \to F \quad & \text{defined by} \\ & (x, f) \mapsto \Phi_{ji}(x)(f). \end{aligned}$$

Denoting the group of homeomorphisms of F as $Homeo(F)$, we reconsider a transition map $\Phi_{ji} : U_i \cap U_j \to Homeo(F)$ as a function into $Homeo(F)$. In this redefined notion, by continuity of Φ_{ji}, we mean continuity of the above map $\tilde{\Phi}_{ji}$. The family $\{\Phi_{ij}\}$ of transition maps satisfies

(i) $\Phi_{kj} \circ \Phi_{ji} = \Phi_{ki}$ at each x in $U_i \cap U_j \cap U_k$,

(ii) Φ_{ii} is the identity map on F at each $x \in U_i$, and

(iii) Φ_{ji} has Φ_{ij} as its inverse at each x in $U_i \cap U_j$.

A data satisfying these three conditions is called a **cocycle** and (i) is called **cocycle condition**. In case of example 4.3, the family of transition maps is isomorphic to $\mathbb{Z}_2$.

Example 4.4. Let K denote the Klein bottle obtained as the quotient space from $[0,1] \times [0,1]$ by the identifications $(s,0) \sim (s,1)$ for $s \in [0,1]$ and $(0,t) \sim (1, 1-t)$ for $t \in [0,1]$. Let C be the subspace $\{[(s,0)] \mid s \in [0,1]\}$ and $p : K \to C$ be the canonical projection map. Fiber of each point $[(s,0)]$ under this map is $\{[(s,t)] \mid t \in [0,1]\}$. Since $(s,0)$ and $(s,1)$ are identified together, the fiber of p is, in fact, S^1. The trivializations are given by

$$\Phi_1 : p^{-1}(U_1) \to U_1 \times S^1 \text{ where}$$
$$U_1 = \{[(s,0)] \mid s \in [0, 1/2) \cup (1/2, 1]\} \ ,$$
$$[(s,t)] \mapsto \big([(s,0)], e^{2\pi i t}\big) \text{ for } s \in [0, 1/2) \text{ and } [(s,t)] \mapsto \big([(s,0)], e^{-2\pi i t}\big) \text{ for } s \in (1/2, 1].$$

$$\begin{gathered}\Phi_2 : p^{-1}(U_2) \to U_2 \times S^2 \text{ where}\\ U_2 = \{[(s,0)] \mid s \in (0,1)\} \text{ and}\\ [(s,t)] \mapsto \left([(s,0)], e^{2\pi i t}\right).\end{gathered}$$

The family of transition maps, here too, is isomorphic to $\mathbb{Z}_2$.

Of special interest is when the maps Φ_{ji} parametrize a special class of homeomorphisms of the fiber F. For instance, if the fiber is a vector space, we would like to have Φ_{ji} at each $x \in U_i \cap U_j$ to be a linear isomorphism of the fiber. In the next section, we will deal with the special case of the fiber being a group and the maps Φ_{ji} parametrizing translation maps of this group.

4.2 Principal G-bundles

Let G be a topological group. By default, we will consider right G-actions and henceforth will refer to them as G-actions.

Definition 4.5. *A topological space X is called a G-space if there exists a continuous group action $X \times G \to X$.*

Example 4.6. Let F be a G-space and B be a topological space. The product space $B \times F$ can be considered as a G-space with G-action given by $(b,f)g = (b,fg)$ for $b \in B$, $f \in F$ and $g \in G$.

Definition 4.7. *Let X and Y be G-spaces. A continuous map $f : X \to Y$ is called a G-morphism if $f(xg) = f(x)g$ for $x \in X$ and $g \in G$.*

G-morphisms, therefore, are natural maps to be considered between G-spaces.

Definition 4.8. *Let G be a topological group. Let $p : E \to B$ be a fiber bundle with fiber F that satisfies the following properties.*

(i) The total space E is a G-space with the underlying G-action preserving fibers, that is, $p(xg) = p(x)$ for $x \in E$ and $g \in G$. Considering $F \hookrightarrow G$, the fiber is also a G-space.

(ii) There exists a cover $\{U\}$ of base space B with local trivializations $\Phi_U : p^{-1}(U) \to U \times F$ that are G-morphisms.

Then the fiber bundle p is called a ***principal*** *G**-bundle**.*

In property (ii), the product $U \times F$ is a G-space with respect to the G-action $(b, f)g = (b, fg)$ for $b \in U, f \in F$ and $g \in G$. Since the G-morphisms Φ_U are local trivializations, the group G acts freely and transitively on F. Thus the fiber F is homeomorphic to G.

Example 4.9. The projection map $p : B \times G \to B$ is called the **trivial principal** G**-bundle** over B. The G-action on $B \times G$ is given as $(b, h)g = (b, hg)$ for $b \in B$ and $h, g \in G$. The identity map $B \times G \to B \times G$ is a global trivialization that is a G-morphism.

Property (ii) of definition 4.8, in view of the above example, says that a principal G-bundle is locally the trivial principal G-bundle over the base space.

Example 4.10. Consider the n-dimensional real projective space $\mathbb{R}P^n$ obtained as the quotient space of S^n under the identifications $x \sim -x$ for $x \in S^n$. Let $p : S^n \to \mathbb{R}P^n$ be the projection map that sends each point of S^n to its equivalence class under the above identifications. This map is a covering map. Indeed, if $[(x_1, \ldots, x_n)] \in \mathbb{R}P^n$ with $x_i \neq 0$ for some $1 \leq i \leq n$, then the image set $p\left(\{(x_1, \ldots, x_n) \in S^n \mid x_i > 0\}\right)$ is an evenly covered neighborhood containing $[(x_1, \ldots, x_n)]$. Therefore p is a fiber bundle whose fiber is homeomorphic to $\mathbb{Z}_2$. Further, the local trivializations of p given in example 4.2 are $\mathbb{Z}_2$-morphisms. Thus p is a principal $\mathbb{Z}_2$-bundle.

Example 4.11. A covering space $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is normal if and only if the group G of deck transformations of p acts transitively on the fiber of the base point x_0. Therefore, p becomes a principal G-bundle with G-morphic local trivializations as given in example 4.2. Consequently, a map $p : \widetilde{X} \to X$ of path connected spaces is a principal $\mathbb{Z}_2$ bundle if and only if it is a connected covering map of degree two. Note that the above example is a double covering. Also, a universal covering map $p : \widetilde{X} \to X$ is a principal $\pi_1(X, x_0)$-bundle for $x_0 \in X$.

Example 4.12. Consider the n-dimensional complex projective space $\mathbb{C}P^n$ obtained as the quotient space from the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ via the identifications $x \sim \lambda x$ for $\lambda \in S^1 \subset \mathbb{C}$. Let $p : S^{2n+1} \to \mathbb{C}P^n$ be the projection map that sends each point $(z_0, \ldots, z_n)$ of S^{2n+1} to its equivalence class $[z_0 : \cdots : z_n]$ under the above identifications. Then p is a principal S^1-bundle. To see this, consider the sets $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_i \neq 0\}$ for $i = 0, \ldots, n$. Since $p^{-1}(U_i) = \{(z_0, \ldots, z_n) \in S^{2n+1} \mid z_i \neq 0\}$ is open for each i , the collection $\{U_i\}_{i=0}^n$ is an open cover of $\mathbb{C}P^n$. Define $\Phi_i : p^{-1}(U_i) \to U_i \times S^1$ as

$$(z_0, \dots, z_n) \mapsto \left([z_0 : \cdots : z_n], \frac{z_i}{|z_i|} \right).$$

To see that Φ_i is a homeomorphism, define the inverse map as

$$([z_0 : \cdots : z_n], \lambda) \mapsto \lambda |z_i| \left(\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i} \right).$$

Certainly, the maps Φ_i are S^1-morphisms.

4.3 Bundle morphisms

In this section, we will consider the natural maps between fiber bundles.

Definition 4.13. *A **bundle morphism** between fiber bundles $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ is a continuous map $\tilde{f} : E_1 \to E_2$ such that there exists a continuous map $f : B_1 \to B_2$ satisfying $p_2 \circ \tilde{f} = f \circ p_1$, that is, the following diagram commutes.*

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

*The bundle morphism $\tilde{f}$ is called a **bundle isomorphism** if $\tilde{f}$ is a homeomorphism and $(\tilde{f})^{-1}$ is a bundle morphism between p_2 and p_1.*

For fiber bundles with same base space, we have the following notion of bundle morphism.

Definition 4.14. *A **bundle morphism over** B between fiber bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ is a continuous map $\tilde{f} : E_1 \to E_2$ such that $p_2 \circ \tilde{f} = p_1$, that is, the following diagram commutes.*

$$\begin{tikzcd} E_1 \arrow[rr, "\tilde{f}"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

*The bundle morphism $\tilde{f}$ is called a **bundle isomorphism over** B if $\tilde{f}$ is a homeomorphism and $(\tilde{f})^{-1}$ is a bundle morphism over B between p_2 and p_1.*

Example 4.15. Let $p : E \to B$ and $\tilde{p} : \tilde{E} \to \tilde{B}$ be fiber bundles such that E and B are subspaces of $\tilde{E}$ and $\tilde{B}$ respectively, and $p = \tilde{p}|_E : E \to B$. The bundle morphism between p and $\tilde{p}$ is the inclusion map $E \hookrightarrow \tilde{E}$.

Example 4.16. A deck transformation between two connected covering maps $p_1 : (\widetilde{X_1}, \tilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X_2}, \tilde{x}_2) \to (X, x_0)$ is a bundle isomorphism over X.

Definition 4.17. *A bundle morphism $\tilde{f}$ between principal G-bundles $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ is called a **principal G-bundle morphism** if $\tilde{f}$ is a G-morphism. Further, if $\tilde{f}$ is a homeomorphism and $(\tilde{f})^{-1}$ is a principal G-bundle morphism between p_2 and p_1, then $\tilde{f}$ is called a principal G-bundle isomorphism.*

Definition 4.18. *Let $\tilde{f}$ be a bundle morphism over B between principal G-bundles $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$. Then $\tilde{f}$ is called a **principal G-bundle morphism over** B if $\tilde{f}$ is a G-morphism. Further, if $\tilde{f}$ is a homeomorphism and $(\tilde{f})^{-1}$ is a principal G-bundle morphism over B between p_2 and p_1, then $\tilde{f}$ is called a principal G-bundle isomorphism over B.*

Definition 4.19. *Let $f : X \to B$ be a continuous map and let $p : E \to B$ be a fiber bundle. The **pullback bundle** or the **induced bundle** of p under f is the fiber bundle $\mathrm{pr}_X : f^*E \to X$, where the total space f^*E is the subspace $\{(x, e) \in X \times E \mid f(x) = p(e)\}$ and pr_X is the projection map onto X.*

Indeed, the pullback bundle is a fiber bundle. Suppose the fiber of p is F. The fiber of pr_X over $x \in X$ is $\{x\} \times p^{-1}(f(x))$, which is homeomorphic to F. Let $\Phi = (\Phi_1, \Phi_2) : p^{-1}(U) \to U \times F$ be a local trivialization of p. Then the induced map $\Phi^* : \{(x, e) \in f^*E \mid x \in f^{-1}(U)\} \to f^{-1}(U) \times F$ defined as $(x, e) \mapsto (x, \Phi_2(e))$ is a local trivialization of pr_X.

Example 4.20. Let $\iota : A \hookrightarrow B$ be an inclusion map of spaces and $p : E \to B$ be a fiber bundle. Then the induced fiber bundle is $\mathrm{pr}_A : \iota^*E \to A$ where $\iota^*E = \{(a, e) \in A \times E \mid a = f(e)\}$. There is a canonical bundle morphism $\tilde{\iota}$ such that the following diagram commutes.

$$\begin{tikzcd}
\iota^*E \arrow[r, "\tilde{\iota}"] \arrow[d, "pr_A"'] & E \arrow[d, "p"] \\
A \arrow[r, "\iota"] & B
\end{tikzcd}$$

Let $p : E \to B$ be a principal G-bundle and let $f : X \to B$ be a continuous map of topological spaces. Then the pullback bundle $\mathrm{pr}_X : f^*E \to X$ of p is a principal G-bundle. For this, define the group action on f^*E by $(x, e)g \mapsto (x, eg)$ for $(x, e) \in f^*E$ and $g \in G$. This is a well-defined group action that makes f^*E into a G-space. Let $\Phi = (\Phi_1, \Phi_2) : p^{-1}(U) \to U \times F$ be a G-morphic local trivialization of the principal bundle p. Since Φ is a G-morphism, $\Phi_2(eg) = \Phi_2(e)g$ for $e \in p^{-1}(U)$ and $g \in G$. Then the induced map $\Phi^* : \{(x, e) \in f^*E \mid x \in f^{-1}(U)\} \to f^{-1}(U) \times F$ defined as $(x, e) \mapsto (x, \Phi_2(e))$ is a local trivialization of pr_X that is a G-morphism.

Example 4.21. Consider the principal $\mathbb{Z}_2$-bundle $S^n \to \mathbb{R}P^n$, the universal covering map $\mathbb{R} \to S^1$ and the principal S^1-bundle $S^{2n+1} \to \mathbb{C}P^n$. Let X be a topological space. Then continuous maps $X \to \mathbb{R}P^n$, $X \to S^1$ and $X \to \mathbb{C}P^n$ result in respective pullback principal G-bundles.

Chapter 5

Construction of $K(G, 1)$ spaces

In this chapter, we will use construction of universal G-bundles in [Milnor, 1956b] to obtain a space whose first homotopy group is G and higher homotopy groups are trivial. The construction of universal G-bundles will rely on the notions of join of spaces and principal G-bundles. Henceforth, join of spaces will refer to the join of definition 3.13, unless stated otherwise.

5.1 Construction of Universal Bundles

Definition 5.1. *A principal G-bundle with the total space $(n-1)$-connected is called an n-universal G-bundle. A principal G-bundle with the total space ∞-connected is called an ∞-universal G-bundle.*

Let G be a topological group. Denote the join $G \circ \cdots \circ G$ of $(n+1)$ copies of G by E_nG. Denote the join of countably infinite copies of G by EG. Each E_nG is a closed subspace of EG. Define the right translations $R_n : E_nG \times G \to E_nG$ and $R : EG \times G \to EG$ by

$$R_n(t_0g_0 \oplus \cdots \oplus t_ng_n, g) = t_0(g_0g) \oplus \cdots \oplus t_n(g_ng) \text{ and}$$
$$R(\underset{i\in\mathbb{N}_0}{\oplus} t_ig_i, g) = \underset{i\in\mathbb{N}_0}{\oplus} t_i(g_ig).$$

Observe that the restricted map $R : E_nG \times G \to E_nG$ is equal to R_n. Let B_nG and BG be the G-orbit spaces obtained from E_nG and EG respectively. Let $q_n : E_nG \to B_nG$ and $q : EG \to BG$ be the associated quotient maps that project a point to its equivalence class. Each B_nG is a closed subspace of BG. The space BG is called a **classifying space** of the group G.

Lemma 5.2. *The maps on* B_nG

$$B_n\theta_j : [t_0g_0 \oplus \cdots \oplus t_ng_n] \mapsto t_j \quad \textit{and}$$
$$B_n\chi_{ij} : [t_0g_0 \oplus \cdots \oplus t_ng_n] \mapsto g_jg_i^{-1}$$

for $i, j = 0, \ldots, n$ *are continuous on their appropriate domains. The maps on* BG

$$B\theta_j : [\bigoplus_{i\in\mathbb{N}_0} t_ig_i] \mapsto t_j \quad \textit{and}$$
$$B\chi_{ij} : [\bigoplus_{i\in\mathbb{N}_0} t_ig_i] \mapsto g_jg_i^{-1}$$

for $i, j \in \mathbb{N}_0$ *are continuous on their appropriate domains.*

Proof. We note that the map $B_n\theta_j$ is induced by the coordinate map θ_j, that is, the following diagram commutes.

$$\begin{tikzcd}
E_nG \arrow[r, "\theta_j"] \arrow[d, "q_n"] & I \\
B_nG \arrow[ur, "B_n\theta_j"'] &
\end{tikzcd}$$

Continuity of $B_n\theta_j$ follows from the fact that θ_j is an open map. The map $B_n\chi_{ij}$ is defined at those points with t_i and t_j non-zero. For $i, j = 0, \ldots, n$, denote the product of the continuous maps χ_i and $t_0g_0 \oplus \cdots \oplus t_ng_n \mapsto g_j^{-1}$ by f_{ij}. The map f_{ij} is defined on the intersection of domains of its factors. Thus the map $B_n\chi_{ij}$ is induced by the map f_{ij} , that is the following diagram commutes.

$$\begin{tikzcd}
E_nG \arrow[r, "f_{ij}"] \arrow[d, "q_n"] & G \\
B_nG \arrow[ur, "B_n\chi_{ij}"'] &
\end{tikzcd}$$

We proceed similarly for the maps $B\theta_j$ and $B\chi_{ij}$, for $i, j \in \mathbb{N}_0$. ■

Lemma 5.3. *The spaces* E_nG *and* EG *are* G*-spaces. The underlying group actions preserve fibers under the respective quotient maps* q_n *and* q.

Proof. We need to show that R_n and R are continuous maps. The coordinates of the map R_n are

$$\theta_j \circ R_n : (t_0g_0 \oplus \cdots \oplus t_ng_n, g) \mapsto t_j \quad \text{for } j = 0, \dots, n \quad \text{and}$$
$$\chi_j \circ R_n : (t_0g_0 \oplus \cdots \oplus t_ng_n, g) \mapsto g_jg \quad \text{for } j = 0, \dots, n.$$

The coordinates of the map R are

$$\theta_j \circ R : (\oplus_i t_ig_i, g) \mapsto t_j \quad \text{for } j \in \mathbb{N}_0 \quad \text{and}$$
$$\chi_j \circ R : (\oplus_i t_ig_i, g) \mapsto g_jg \quad \text{for } j \in \mathbb{N}_0.$$

The coordinate $\theta_j \circ R_n$ is the composition $(t_0g_0 \oplus \cdots \oplus t_ng_n, g) \mapsto t_0g_0 \oplus \cdots \oplus t_ng_n \mapsto t_j$ of continuous maps. The coordinate $\chi_j \circ R_n$ is defined at those points of E_nG that have t_j non-zero. Hence χ_j is the product of the continuous maps $(t_0g_0 \oplus \cdots \oplus t_ng_n, g) \mapsto t_0g_0 \oplus \cdots \oplus t_ng_n \mapsto g_j$ and $(t_0g_0 \oplus \cdots \oplus t_ng_n, g) \mapsto g$. Similarly, it can be checked that the coordinates of R are continuous. The second part of the theorem follows from the definitions of q_n and q. ■

Lemma 5.4. *The map* $q_n : E_nG \to B_nG$ *is an* $(n-1)$*-universal* G*-bundle.*

Proof. We need to exhibit local trivializations of B_nG that are G-morphisms. Let $U_i = \{[t_0g_0 \oplus \cdots \oplus t_ng_n] \in B_nG \mid t_i \neq 0\}$ for $i = 0, \dots, n+1$. Since $q_n^{-1}(U_i)$ is open for each i, the collection $\{U_i\}_{i=0}^n$ is an open cover of B_nG. Define the local trivializations $\Phi_i : U_i \times G \to q_n^{-1}(U_i)$ by

$$\Phi_i\left([t_0g_0 \oplus \cdots \oplus t_ng_n], g\right) = t_0(g_0g_i^{-1}g) \oplus \cdots \oplus t_n(g_ng_i^{-1}g)$$

for $i = 0, \dots, n$. The maps Φ_i are well-defined. The coordinates of Φ_i are the maps

$$\theta_j \circ \Phi_i : ([t_0g_0 \oplus \cdots \oplus t_ng_n], g) \mapsto t_j \text{ and}$$
$$\chi_j \circ \Phi_i : ([t_0g_0 \oplus \cdots \oplus t_ng_n], g) \mapsto g_jg_i^{-1}g$$

for $j = 0, \dots, n$. It is clear from lemma 5.2 that the coordinates of Φ_i are continuous. To show that Φ_i is a homeomorphism, consider the inverse map $\Phi_i^{-1} : q_n^{-1}(U_i) \to U_i \times G$ defined by

$$t_0g_0 \oplus \cdots \oplus t_ng_n \mapsto ([t_0g_0 \oplus \cdots \oplus t_ng_n], g_i).$$

The inverse map Φ_i^{-1} is continuous as each of its components q_n and χ_i are continuous on $q_n^{-1}(U_i)$. Evidently, the maps Φ_i are G-morphisms. Lemma 3.31 gives that the total space E_nG is $(n-1)$-connected. ■

Now we will construct an ∞-universal G-bundle for a given topological group G that will show that a classifying space of the group G exists.

Lemma 5.5. *The map $q : EG \to BG$ is an ∞-universal G-bundle.*

Proof. We proceed as in the proof of the previous theorem.
Let $V_i = \{[\oplus_j t_j g_j] \in BG \mid t_i \neq 0\}$ for $i \in \mathbb{N}_0$. Since each $q^{-1}(V_i)$ is open, the collection $\{V_i\}_{i \in \mathbb{N}_0}$ is an open cover of BG. Define the local trivializations $\Psi_i : V_i \times G \to q^{-1}(V_i)$ by

$$\Psi_i\left([\oplus_j t_j g_j], g\right) = \oplus_j t_j (g_j g_i^{-1} g)$$

for $i \in \mathbb{N}_0$. The coordinates of each of the maps Ψ_i are continuous by lemma 5.2. To see that Ψ_i is continuous, consider the inverse map $\Psi_i^{-1} : q^{-1}(V_i) \to V_i \times G$ defined by

$$\oplus_j t_j g_j \mapsto \left([\oplus_j t_j g_j], g_i\right).$$

It is easy to see that the coordinates of Ψ_i^{-1} are continuous. The maps Ψ_i are G-morphisms and it follows from lemma 3.32 that EG is ∞-connected. ■

5.2 Construction of $K(G,1)$ spaces

Definition 5.6. *Let G be a group with discrete topology. A path connected space whose fundamental group is G and all other homotopy groups trivial is called a **$K(G,1)$ space**.*

We have the following result from [Hatcher, 2002] (p. 342).

Lemma 5.7. *A covering map $(\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ induces isomorphisms $p_* : \pi_n(\widetilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ for $n \geq 2$.*

Proof. Let $n \geq 2$ and $(S^n, s_0) \to (X, x_0)$ be a continuous map. Theorem A.31 gives a lift of this map under p because $\pi_1(S^n, *)$ is trivial for $n \geq 2$. This shows that p_* is surjective. Injectivity of p_* is ensured by theorem A.26. ■

Finally, we have our required result.

Theorem 5.8. *Let G be a group with discrete topology. Then there exists a $K(G,1)$ space.*

Proof. Consider the construction of an ∞-universal G-bundle $q : EG \to BG$ as in lemma 5.5. Since the group G is discrete, the map q is a universal covering map. This means that the base space BG has a fundamental group isomorphic to G. Since the total space has all homotopy groups trivial, it follows from the above lemma that homotopy groups π_n, for $n \geq 2$, of BG are trivial. Therefore, the base space BG is a $K(G, 1)$ space. ■

Also, we have the following corollary as a consequence of lemma 5.7.

Corollary 5.9. *Let G be a group with discrete topology. A path connected space with fundamental group G and a contractible universal covering space is a* $K(G, 1)$ *space.* ■

5.3 Uniqueness of $K(G, 1)$ spaces

Uniqueness of $K(G, 1)$ spaces is guaranteed by the following technical lemma, quoted from [Hatcher, 2002] (p. 90), whose proof we skip.

Lemma 5.10. *Let* G *be a group with discrete topology. Let* X *be a connected* CW*-complex and let* Y *be a* $K(G, 1)$ *space. Then every homomorphism* $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ *is induced by a continuous map* $f : (X, x_0) \to (Y, y_0)$*. If* $g : (X, x_0) \to (Y, y_0)$ *is another continuous map that induces this homomorphism, then there exists a homotopy* $X \times I \to Y$ *between* f *and* g *that fixes* (x_0, t) *for* $t \in I$*.*

Theorem 5.11. *Let* G *be a group with discrete topology. Then all* $K(G, 1)$ *spaces that are* CW*-complexes are homotopy equivalent to each other.*

Proof. Let (X, x_0) and (Y, y_0) be two $K(G, 1)$ spaces that are CW-complexes. Then the isomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ is induced by a continuous map $f : (X, x_0) \to (Y, y_0)$ and the isomorphism $\pi_1(Y, y_0) \to \pi_1(X, x_0)$ is induced by a continuous map $g : (Y, y_0) \to (X, x_0)$. This means, the composition $f \circ g$ is homotopic to the identity map on X because it induces the identity isomorphism $\pi_1(X, x_0) \to \pi_1(X, x_0)$. Similarly, the composition $g \circ f$ is homotopic to the identity map on Y. ■

Now we give a CW-complex structure on the total spaces of universal G-bundles constructed in the first section. The particular case of G being discrete follows.

Definition 5.12. *Let G be a topological group. If G is a countable CW-complex with the multiplication map $G \times G \to G$ and the inverse map $G \to G$ as cellular maps, then G is called a CW-group.*

We restrict ourselves to the class of countable CW-groups G. The group G is taken to be a countable CW-complex so that the product topology and the weak topology of CW-complex structure on $G \times G$ agree. Let ε denote the identity element of G. Then the condition on the multiplication map and the inverse map to be cellular maps forces ε to be in the 0-skeleton G^0 of G. With abuse of notation, let ε also denote the singleton containing ε.

We have the following from [Milnor, 1956b] (p. 435).

Theorem 5.13. *Let G be a countable CW-group. Then there exists a countable CW-complex structure on the spaces E_nG, B_nG, EG and BG such that the group actions of E_nG and EG are cellular maps.*

Proof. We will prove the result for E_nG and B_nG by induction on n. The CW-complex structure on $E_0G = G$ is the same as that of G. Now consider E_nG as $E_{n-1} \circ G$. This can be done because of associativity of joins. The induction hypothesis is that $E_{n-1}G$ is a countable CW-complex with the group action $R_{n-1} : E_{n-1}G \times G \to E_{n-1}$ being a cellular map. Let τ denote a generic cell of $E_{n-1}G$ and its characteristic map be Φ_τ. Let σ denote a generic cell of G with the characteristic map Φ_σ. Then $(\tau \circ \varepsilon)\sigma$ is the set of all right translates $R_n(tx \oplus (1-t)\varepsilon, g)$ for $x \in \tau, g \in \sigma$ and $t \in [0,1]$. The cell τ is considered as a cell of E_nG by extending the codomain of the characteristic map Φ_τ to E_nG. Similarly the cell σ is seen as a cell of E_nG. If τ is an i-cell and σ is a j-cell, then $(\tau \circ \varepsilon)\sigma$ is an $(i+j+1)$-cell of E_nG. Indeed, the cell $(\tau \circ \varepsilon)\varepsilon$ is an $(i+1)$-cell with the characteristic map $\Phi_{\tau\circ\varepsilon} : D^{i+1} \to E_nG$ defined by

$$(u, t) \mapsto t\Phi_\tau(u) \oplus (1-t)\varepsilon$$

for $u \in D^i$ and $t \in D^1$. Then the characteristic map Φ required to consider $(\tau \circ \varepsilon)\sigma$ as a cell of E_nG is the composition $R_n \circ (\Phi_{\tau\circ\varepsilon} \times \Phi_\sigma)$.
We observe that an arbitrary point $tx \oplus (1-t)g$ of $E_{n-1}G \circ G$ is, in fact, of the form $R_n(t(R_{n-1}(x, g^{-1})) \oplus (1-t)\varepsilon, g)$. Here if x is in some i-cell of $E_{n-1}G$, then $R_{n-1}(x, g^{-1})$ is in some i-cell too, because of the induction hypothesis that R_{n-1} is cellular. Therefore, the above characteristic maps Φ along with Φ_τ and Φ_σ encompass all the points of E_nG. Since the multiplication map of G and the

right translation R_{n-1} are cellular maps, it follows that R_n is a cellular map. Evidently E_nG is a countable CW-complex.

Now consider $B_nG = q_n(E_{n-1}G \circ G)$. Let τ' be a generic cell of $B_{n-1}G$ with the characteristic map $\Phi_{\tau'}$. The cells σ of G in E_nG get identified to the point $q_n(\sigma)$ by the map q_n. Further, the cells $(\tau \circ \varepsilon)\sigma$ of E_nG get identified to the cell $q_n((\tau \circ \varepsilon)\varepsilon)$ where τ is a generic cell of E_nG. Therefore the cells of B_nG are τ', the 0-cell $q_n(\sigma)$, and the cells $q_n((\tau \circ \varepsilon)\varepsilon)$. If τ is an i-cell, then $q_n((\tau \circ \varepsilon)\varepsilon)$ is an $(i+1)$-cell. Denote the characteristic map of $(\tau \circ \varepsilon)\varepsilon$, considered as a cell of E_nG, by $\Phi_{\tau\circ\varepsilon}$. The map q_n is the identity map on the cell $(\tau \circ \varepsilon)\varepsilon$. Hence the characteristic maps of B_nG are $\Phi_{\tau'}$, $q_n \circ \Phi_{\tau\circ\varepsilon}$ and the inclusion map of $q_n(\sigma)$.

Finally, the space EG is a CW-complex structure with weak topology with respect to the subspaces E_nG. Similarly, BG is given the weak topology with respect to the subspaces B_nG. ■

It can be shown that the right translations R_n and R are continuous with respect to weak topology on E_nG and EG respectively ([Milnor, 1956b] p. 435). Also, the maps q_n and q are universal G-bundles in this case. However, we will consider the case of G being a group with discrete topology.

Theorem 5.14. *Let G be a group with discrete topology. Then the join topology and weak topology on E_nG agree and the quotient topology and weak topology on B_nG agree. Analogous results hold true for EG and BG.*

Proof. Consider $E_nG = E_{n-1}G \circ G$. Denote the coordinate functions defined on $E_{n-1}G \circ G$ onto I, $E_{n-1}G$ and G as θ, χ_1 and χ_2 respectively. Let V be open in I. Then $\theta^{-1}(V) = \{tx \oplus (1-t)g \mid x \in E_{n-1}G, g \in G, t \in V\} = \cup_g\{tx \oplus (1-t)\varepsilon \mid x \in E_{n-1}G, t \in V\}$ is union of open sets in the weak topology of E_nG. Let W be open in $E_{n-1}G$. Then $\chi^{-1}(W) = \cup_g\{tx \oplus (1-t)\varepsilon \mid x \in W, t \in I\}$ is open in weak topology. Finally $\chi^{-1}(g) = \{tx \oplus (1-t)g \mid x \in E_{n-1}G, t \in I\} = \cup_g\{tx \oplus (1-t)\varepsilon \mid x \in E_{n-1}G, t \in I\}$ is open in weak topology.

Let X_i denote the i^{th} copy of G, for $i \in \mathbb{N}_0$. Then, the joins E_nG and $J(G_i)_{i \leq n}$ are homeomorphic by theorem 3.28. Considering E_nG with product topology, it is possible to show that open sets in weak topology of E_nG are open in product topology using the standard technique of constructing product neighborhoods in CW-complexes; refer [Hatcher, 2002] p. 522.

Since q_n is a local homeomorphism, we have our result for B_nG as well. Proceed similarly for EG and BG.

■

5.4 Examples of $K(G,1)$ spaces

Given a group G with discrete topology, we can find a $K(G,1)$ space. Each such space is called a model for $K(G,1)$, and is unique up to homotopy type, if the model is a CW-complex. The construction of ∞-universal bundle gives a particular model of $K(G,1)$. There are, in fact, other ways of constructing $K(G,1)$ spaces. A simplicial model can be found in [Hatcher, 2002] (p. 89). In practice, there could be a more effective model for $K(G,1)$ that might not be provided by these constructions.

Example 5.15. Let $G = \mathbb{Z}_2$. Then E^nG is S^{n-1} and B^nG is $\mathbb{R}P^{n-1}$. Therefore, the total space EG is the infinite sphere S^∞ and the base space BG is $\mathbb{R}P^\infty$. Therefore $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2,1)$ that is unique up to homotopy type. This could have been obtained from other results as well. It was shown in 2.21 that S^∞ is contractible. Since S^∞ is a double cover of $\mathbb{R}P^\infty$, it follows that $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2,1)$ space from 5.7. However, it needs to be checked that the CW-complex structure on $\mathbb{R}P^\infty$ in the former case is same as the one in the latter case. Indeed, this is true because the n-skeletons of both structures are homeomorphic.

Example 5.16. Let G be a free group with discrete topology. Indeed BG is a $K(G,1)$ space but there is a more effective model. In chapter 1, it was shown that there exists a connected graph (X,x_0) whose $\pi_1(X,x_0)$ is isomorphic to G. By theoremA.35, there exists a universal cover $p : (\widetilde{X},\tilde{x}_0) \to (X,x_0)$. Theorem 1.23 gives that $(\widetilde{X},\tilde{x}_0)$ is a graph. As $(\widetilde{X},\tilde{x}_0)$ is simply connected, it is a tree by corollary 1.21. Thus X is a $K(G,1)$ space by corollary 5.9. If G is a countable free group, then BG and X are homotopy equivalent.

Example 5.17. Let $G = \mathbb{Z}$. Then S^1 is a $K(\mathbb{Z},1)$ by corollary 5.9. Certainly $B\mathbb{Z}$ is a $K(\mathbb{Z},1)$, albeit an intractable one. However, the space $B\mathbb{Z}$ is homotopy equivalent to S^1.

Example 5.18. Let G be S^1 with discrete topology. Then, as sets, the total space E^nG is the sphere S^{2n-1} and EG is S^∞. The base space BG is a $K(S^1,1)$. However, we cannot comment on the uniqueness of this space as S^1 is not a countable CW-group.

5.5 Further notes and references

The assignment $G \mapsto BG$ is a functor from the category of topological groups to the category of topological spaces. The classifying space BG is primarily important because there is a bijection between the homotopy classes of maps $X \to BG$ and isomorphism classes of principal G-bundles over a paracompact Hausdorff space X. We have seen that given a map $f : X \to BG$, the pullback bundle of f is a principal G-bundle over X. The correspondence says that given a principal G-bundle p over X, there exists a map $\phi : X \to BG$ whose pullback is isomorphic to the given bundle over X. This classifying map ϕ is unique upto homotopy. Refer [Husemoller, 1994] for further details.

The S^1-action on the infinite sphere S^∞, as seen in 2.23 is, in fact, the universal S^1-bundle obtained via Milnor's construction. The S^1-orbit space, called as the infinite-dimensional complex projective space $\mathbb{C}P^\infty$, is the classifying space of S^1. Thus $\mathbb{C}P^\infty$ classifies the principal S^1-bundles over a paracompact Hausdorff space X.

Appendix A

Background material

A.1 Quotient spaces

This section is compiled from [Armstrong, 1983] and [Munkres, 2000].

Definition A.1. *Let X be a topological space, Y be a set and $q : X \to Y$ be a surjective map. Then q is called a* ***quotient map*** *if Y has the largest topology for which q is continuous. This topology on Y is called the* ***quotient topology*** *with respect to q.*

Therefore, a subset A of Y is in the quotient topology of Y with respect to q if and only if $q^{-1}(A)$ is open in X. If the map q is clear from the context, the topology on Y is simply referred to as the quotient topology.

Definition A.2. *Let X be a topological space with a partition, that is, the space X can be written as the disjoint union of subsets X_α for $\alpha \in \Lambda$. Denote $\{X_\alpha\}_\alpha$ by X^* and let $q : X \to X^*$ be the projection map sending each point to the subset X_α containing it. The* ***quotient space*** *of X with respect to this partition is defined to be the space X^* with the quotient topology.*

Let an equivalence relation $\sim$ generate the partition on X. The quotient space is denoted as ${}^{X}\!/_{\sim}$ in such a case. If the equivalence relation $\sim$ is induced by a group G acting on X, then ${}^{X}\!/_{G}$ is used to denote the quotient space. In this case, ${}^{X}\!/_{G}$ is called the **G-orbit space** of X. If the equivalence relation $\sim$ is induced by identifying all points of a subspace A of X, then ${}^{X}\!/_{A}$ denotes the quotient space. In the last case, we say that the quotient space ${}^{X}\!/_{A}$ is obtained by collapsing the subspace A.

Definition A.3. *Let $f : X \to Y$ be a function of sets. Then the set $f^{-1}(y)$ is called the* ***fiber*** *of f over y, for $y \in Y$.*

We quote the following theorem from [Munkres, 2000] that is useful for checking continuity of a map defined on a quotient space.

Theorem A.4. *Let $q : X \to Y$ be a quotient map. Let Z be a topological space and let $f : X \to Z$ be a function that is constant on each fiber $q^{-1}(y)$, for $y \in Y$. Then f induces a map $h : Y \to Z$ such that $h \circ q = f$, that is, the following diagram commutes.*

$$\begin{tikzcd}
X \arrow[d, "q"] \arrow[dr, "f"] & \\
Y \arrow[r, "h"'] & Z
\end{tikzcd}$$

The map h is continuous if and only if f is continuous. The map h is a homeomorphism if and only if f is a quotient map.

Any quotient map $q : X \to Y$ partitions X into the fibers $q^{-1}(y)$ for $y \in Y$. Let X^* denote the collection of these fibers with quotient topology with respect to the the projection map $p : X \to X^*$ as defined in definition A.2. Then we have the following result from [Armstrong, 1983] as a corollary of the above theorem.

Corollary A.5. *If $q : X \to Y$ is a quotient map of topological spaces, then Y is homeomorphic to X^*.*

A.2 Homotopy and fundamental groups

This section is compiled from [Munkres, 2000] and [Hatcher, 2002].

Definition A.6. *Let $f : X \to Y$ and $g : X \to Y$ be continuous maps of topological spaces. Then f and g are said to be **homotopic** maps if there exists a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. The map H is called a **homotopy** of maps f and g.*

A map $f : X \to Y$ is said to be nullhomotopic if f is homotopic to a constant map $X \to Y$. Homotopy of maps is an equivalence relation (refer [Munkres, 2000]). We call the equivalence class of the continuous map f as the homotopy class of f.

Definition A.7. *Let X be a topological space and let $x_0, x_1 \in X$. A **path** joining x_0 and x_1 in X is a continuous map $f : I \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. If every pair of points in X can be joined by a path then X is called a **path connected** space.*

Definition A.8. *Let X be a topological space. Given $x \in X$ and a neighborhood U of x, if we can find a path connected subset of U containing x, then X is said to be* ***locally path connected****.*

Definition A.9. *Let $f : I \to X$ and $g : I \to X$ be two paths in X joining x_0 and x_1 in X. Then f and g are said to be* ***path homotopic*** *if there exists a homotopy $H : I \times I \to X$ of f and g such that $H(0, t) = x_0$ and $H(1, t) = x_1$. The homotopy H is called a* ***path homotopy*** *of f and g.*

Path homotopy is an equivalence relation (refer [Munkres, 2000]). We call the equivalence class of the path f as the path homotopy class of f and denote it by $[f]$.

Definition A.10. *If $f : I \to X$ is a path such that $f(0) = f(1) = x_0$, then f is called a* ***loop*** *based at $x_0 \in X$.*

Definition A.11. *Let $f : I \to X$ and $g : I \to X$ be two loops based at $x_0 \in X$. The* ***product of loops*** *f and g is the path $f * g : I \to X$ defined by*

$$f * g(t) = \begin{cases} f(2t) & \textit{for } t \in [0, \frac{1}{2}], \\ g(2t - 1) & \textit{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Refer [Munkres, 2000] for the proof of the following results.

Theorem A.12. *The operation $*$ of product of loops based at x_0 in a space X induces a well-defined operation on path homotopy classes of loops based at x_0 in X. We again denote this induced operation by $*$. The set of path homotopy classes of loops based at x_0 is a group with the induced operation $*$. This group is called the* ***fundamental group*** *of X based at x_0, denoted as $\pi_1(X, x_0)$.*

Theorem A.13. *If X is a path connected space, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$ for $x_0, x_1 \in X$.*

Definition A.14. *A path connected space with trivial fundamental group is called a* ***simply connected*** *space.*

If $h : X \to Y$ is a map of sets that sends $x_0 \in X$ to $y_0 \in Y$, then we write this as $h : (X, x_0) \to (Y, y_0)$.

Definition A.15. *Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Then the homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $h_*([\gamma]) = [h \circ \gamma]$ for $\gamma \in \pi_1(X, x_0)$ is called the homomorphism of fundamental groups* ***induced*** *by h at x_0.*

Refer [Munkres, 2000] for the following.

Theorem A.16. *Let* $h : (X, x_0) \to (Y, y_0)$ *be a homeomorphism. Then the homomorphism of fundamental groups induced by* h *at* x_0 *is an isomorphism.*

Definition A.17. *A continuous map* $f : X \to Y$ *of topological spaces is called a* ***homotopy equivalence*** *if there exists a continuous map* $g : Y \to X$ *such that* $f \circ g$ *is homotopic to the identity map of* X *and* $g \circ f$ *is homotopic to the identity map of* Y*.*

Definition A.18. *Topological spaces* X *and* Y *are said to be* ***homotopy equivalent*** *if there exists a homotopy equivalence between* X *and* Y*.*

Homotopy equivalence is an equivalence relation on spaces.

Definition A.19. *A subspace* $A \subset X$ *is said to be a* ***deformation retract*** *of* X *if there exists a homotopy* $H : X \times I \to X$ *such that* $H(x, 0) = x$*,* $H(x, 1) \in A$ *and* $H(a, t) = a$ *for* $x \in X, a \in A, t \in I$*. The homotopy* H *is said to be a* ***deformation retraction*** *of* X *onto* A*.*

Definition A.20. *A space that is homotopy equivalent to the one-point space is said to be* ***contractible****.*

Definition A.21. *Let* X *be a topological space with base point* x_0*. The set of homotopy classes of maps* $(S^n, *) \to (X, x_0)$ *is called the* n***th-homotopy group*** *of* X *based at* x_0*, denoted by* $\pi_n(X, x_0)$*.*

Definition A.22. *Let* n *be a natural number. A topological space which is nonempty, path connected and has first* n *homotopy groups trivial is called an* n***-connected space****. A space which is* n*-connected for each* $n \in \mathbb{N}$ *is said to be* ∞***-connected****.*

Declare, a non-empty space is (-1)-connected. A path connected space is 0-connected.

Refer [Munkres, 2000] for the following.

Theorem A.23. *A homotopy equivalence* $(X, x_0) \to (Y, y_0)$ *induces isomorphisms* $pi_n(X, x_0) \to pi_n(Y, y0)$ *for* $n \in \mathbb{N}$*.*

A.3 Covering space theory

This section is compiled from [Hatcher, 2002] and [Munkres, 2000].

Definition A.24. *Let X and $\widetilde{X}$ be topological spaces and $p : \widetilde{X} \to X$ be a continuous surjective map. The map p is said to be a* ***covering map*** *if for every $x \in X$ we can find an open neighborhood U of x such that $p^{-1}(U)$ can be written as disjoint union $\coprod_{\alpha\in\Lambda} V_\alpha$ of open sets V_α in $\widetilde{X}$ each of which is homeomorphic to U via the map p.*

The space $\widetilde{X}$ is said to be a **covering space** of X. By abuse of terminology, a covering map $p : \widetilde{X} \to X$ will also be called as a covering space. If $\widetilde{X}$ and X are path connected spaces, the map $p : \widetilde{X} \to X$ is called a **connected covering space**.

Definition A.25. *Let X, $\widetilde{X}$ and Y be topological spaces. Let $p : \widetilde{X} \to X$ be a covering map. A* ***lift*** *of a continuous map $f : Y \to X$ under p is defined to be a continuous map $\tilde{f} : Y \to \widetilde{X}$ such that $p \circ \tilde{f} = f$, that is, the following diagram commutes.*

$$\begin{tikzcd}
 & \widetilde{X} \arrow[d, "p"] \\
Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X
\end{tikzcd}$$

The following theorems are from [Hatcher, 2002].

Theorem A.26 (Homotopy lifting property)**.** *Let Y be topological space and let $p : \widetilde{X} \to X$ be a covering space and $f : Y \times I \to X$ be a homotopy. If a map $\tilde{f}_0 : Y \to X$ that is lift of $f\mid_{Y\times\{0\}}$ is given, then there exists a unique homotopy $\tilde{f} : Y \times I \to \widetilde{X}$ that is a lift of f. That is, the following diagram commutes.*

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d, hook, "\iota"] & \widetilde{X} \arrow[d, "p"] \\
Y \times I \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X
\end{tikzcd}$$

Theorem A.27 (Path lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and let $f : I \to X$ be a path. If a point $\tilde{x} \in p^{-1}(f(0))$ is given, then we can find $\tilde{f} : I \to \widetilde{X}$ that is a unique lift of f such that $\tilde{f}(0) = x$.*

That is, the following diagram commutes where ι is onto $\{0\}$ and $\tilde{\iota}$ is onto $\tilde{x}$.

$$\begin{tikzcd}
* \arrow[r, hook, "\tilde{\iota}"] \arrow[d, hook, "\iota"] & \widetilde{X} \arrow[d, "p"] \\
I \arrow[ru, "\tilde{f}"] \arrow[r, "f"'] & X
\end{tikzcd}$$

Theorem A.28. *Let Y be a connected topological space. Let $p : \widetilde{X} \to X$ be a covering space and let $f : Y \to X$ be a continuous map with lifts $\tilde{f}_1 : Y \to \widetilde{X}$ and $\tilde{f}_2 : Y \to \widetilde{X}$ that agree at one point of Y. Then these two lifts are equal at all points in Y.*

Theorem A.29. *Let $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a connected covering map. The induced map $p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\widetilde{X}$ starting at $\tilde{x}_0$ are loops.*

Theorem A.30. *Let $p : \widetilde{X} \to X$ be a connected covering space. The cardinality of $p^{-1}(x)$ is constant for $x \in X$ and is equal to the index of $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.*

Theorem A.31. *Let $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map and let Y be a path connected and locally path connected space. Given a continuous map $f : (Y, y_0) \to (X, x_0)$, there exists a lift $\tilde{f} : (Y, y_0) \to (\widetilde{X}, \tilde{x}_0)$ of f if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$.*

Definition A.32. *Two covering spaces $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are said to be **isomorphic covering spaces** if there exists a homeomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 \circ f = p_1$, that is the following diagram commutes.*

$$\begin{tikzcd}
\widetilde{X}_1 \arrow[rr, "f"] \arrow[rd, "p_1"'] & & \widetilde{X}_2 \arrow[ld, "p_2"] \\
& X &
\end{tikzcd}$$

*The map f is called an **isomorphism of covering spaces** p_1 and p_2.*

Definition A.33. *Let $p : \widetilde{(X)} \to X$ be a covering map. Then an isomorphism of the covering space p with itself is called a**deck transformation** of the covering space $p : \widetilde{X} \to X$.*

Definition A.34. *Let X be a path connected and locally path connected topological space. Given $x \in X$, if we can find a neighborhood U containing x such* *that the inclusion map induced homomorphism* $\pi_1(U, x) \to \pi_1(X, x)$ *is trivial, then* X *is said to be* ***semilocally simply connected****.*

Refer [Hatcher, 2002] for the following.

Theorem A.35. *Let* X *be a path connected, locally path connected and semilocally simply connected. Then the set of base point-preserving isomorphism classes of connected covering spaces* $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ *is in bijective correspondence with the subgroups of* $\pi_1(X, x_0)$*. The correspondence is obtained by mapping the connected covering space* $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ *to the subgroup* $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ *of* $\pi_1(X, x_0)$*. If base points are ignored, then this mapping gives a bijective correspondence between the isomorphism classes of connected covering spaces* $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ *to the conjugacy classes of subgroups of* $\pi_1(X, x_0)$*.*

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