Electromagnetic response of two low dimensional systems: Graphene and Optical Fibre

A thesis submitted for the degree of
DOCTOR OF PHILOSOPHY

by

Vivek M. Vyas

Department of Physical Sciences
Indian Institute of Science Education & Research - Kolkata
Nadia - 741252 INDIA

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To
My Family

ॐ श्री गुरवे नमः

अखण्डमण्डलाकारं व्याप्तं येन चराचरम्

तत्पदं दर्शितं येन तस्मै श्री गुरवे नमः ।

अज्ञानतिमिरान्धस्य ज्ञानाञ्जन शलाकया

चक्षुरुन्मीलितं येन तस्मै श्री गुरवे नमः ।

Declaration

I hereby declare that the material presented in this thesis is, the result of investigations carried out by me in Physical Research Laboratory, Ahmedabad, and Indian Institute of Science Education & Research - Kolkata, under supervision of Prof. Prasanta K. Panigrahi.

The results reported in the thesis are new to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

In keeping with the general practice of reporting scientific observations due acknowledgement has been made whenever the work described is based on the findings of other investigators.

Vivek M. Vyas

Place: Mohanpur

Date:

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Vivek M. Vyas

Synopsis

Mankind has always been curious about nature and naturally available materials. The question what makes different materials different and who decides their properties, has bothered many people from time to time. Many great thinkers also proposed theories and explanations intending to answer this question in a universal manner, but till this date this question is only partially answered. We now understand that the visible matter is made up of atoms, and it is the interplay between them that decides the bulk properties of a material made out them. Atoms interact amongst themselves and arrange in different forms to realise different phases of matter, which although are made up of same ingredients but show different behaviour under external stimulus. Atoms are known to interact with light, but collection of atoms that form a solid, interact with light in a fundamentally different manner. This difference in behaviour allows one to study and differentiate various materials and various phases of the same material. Such a study hopefully brings one closer to aforementioned question. In cases, where one has sufficient knowledge of microscopic working of a given material, it may be possible to give atleast a qualitatively correct theory of electromagnetic response starting from microscopic building blocks. But as one can guess there are many more materials whose microscopic working itself is not clearly known and hence such microscopic approach fails. In some cases, although the microscopic structure is well known but the complexity of the problem of light-matter interaction itself may not allow one to construct such a microscopic theory. In such cases one resorts to phenomenological approach, whereby one tries to construct an effective theory based on broad arguments and experimental evidences. In this thesis, we have tried to understand electromagnetic response of two systems, graphene and optical fibre. For graphene case we follow the former microscopic route whereas for optical fibre the latter one is used.

A century ago, Kamerlingh Onnes discovered the phenomenon of superconductivity, and since then it has been a subject of amazement and puzzle for scientists [1]. A microscopic theory ex-

plaining this phenomenon came almost half a century later from the work of Bardeen, Cooper and Schrieffer (BCS) [1]. It was showed by Anderson and Nambu, amongst others, that the superconductivity is intimately related to the phenomenon of spontaneous symmetry breaking [2, 3]. In spontaneous symmetry breaking, a system realises a vacuum which is not invariant under certain continuous global symmetries of the Hamiltonian, and leads to creation of gapless Nambu-Goldstone modes. As argued by Anderson and others, in case of BCS theory, this gapless Nambu-Goldstone mode is absorbed by photon to make it massive. It must be pointed out that, although photon becomes massive, the theory always maintains gauge invariance and currents are always conserved. Infact, it was shown by Schwinger that gauge invariance and gauge boson mass can coexist [4]. Above is the so called Anderson-Higgs mechanism. While all experimentally observed superconductors display some or the other type of spontaneous symmetry breaking, there is no fundamental principle indicating that superconductivity can only occur via symmetry breaking. One of the main goals of this thesis is to propose a mechanism of superconductivity without spontaneous symmetry breaking in graphene. The proposed mechanism is demonstrated by constructing two simple models, which are not unphysical and hence can be experimentally realised.

Graphene, an atomically thin monolayer of Graphite, consists of carbon atoms connected to one another via a hexagonal network of covalent bonds. It was experimentally isolated in 2004 by Novoselov *et. al.*, and since then it has opened a new way to simulate relativistic physics in condensed matter systems [5]. It was first shown by Wallace that, the valence band touches conduction band at six different points in Brillouin zone, around which electronic dispersion becomes linear [6]. Later Semenoff showed that electronic excitations around these points, which are known as Dirac points, obey Dirac like equation and are relativistic [7]. From symmetry arguments one finds, that only two of these Dirac points are inequivalent, and are often referred to as valleys. So the low energy electronic excitations in Graphene are nothing but two species of massless Dirac fermions defined on a plane moving with Fermi velocity $\sim 10^6\, m/s$. By suitable choice of substrate or by selective doping, a certain discrete symmetry known as sublattice symmetry can be broken, which results in occurrence of a gap in the electronic spectrum due to which Dirac fermions become massive. Being intrinsically two dimensional, these massive Dirac fermions possess non-trivial spin angular momentum apart from fundamental electronic spin. First two chapters of this thesis are devoted to these discussions. It is observed that the presence of this non-trivial spin is responsible for many interesting phenomena that are discussed in two subsequent chapters.

The third chapter deals with two Abelian gauge theory models. In the first model, it is shown that presence of a dynamical Abelian gauge field in Graphene, which couples oppositely to both valley fermions, gives rise to superconductivity. It is observed that this planar Abelian gauge field via virtual fermion loops gets coupled to external electromagnetic field. This coupling is through a topological mixed Chern-Simons term, which is special to three space-time dimensions. It is this unique coupling that paves the way to superconductivity by developing a gauge invariant mass for the external electromagnetic field. Effects like infinite DC conductivity, Meissner effect and persistent currents, all naturally follow from the same. This is in constrast to BCS theory, where Anderson-Higgs mechanism was responsible for developing photon mass. A topological infinite order Berezinskii-Kosterlitz-Thouless phase transition is seen to occur at a certain finite temperature, due to which the Abelian gauge field develops singularities. It is found that these singularities contribute to electromagnetic response of the system, and precisely cancel the superconducting contribution. So, the Berezinskii-Kosterlitz-Thouless phase transition results in a superconductor-insulator transition, which marks loss of superconductivity. Graphene samples obtained in reality are of finite size, with a well defined boundary. It is shown, in case of arm-chair edged boundary, that the boundary supports dissipationless and gapless chiral modes. The presence of mixed Chern-Simons coupling in bulk is seen to play the pivotal role in establishing this bulk-boundary interplay. Infact, it is shown that the theory living on the boundary is identical to Schwinger model on a circle. It is worth emphasizing, that since in the above model superconductivity does not occur because of spontaneous symmetry breaking, there exists no local order parameter characterising ordered/disordered phase.

The second model deals with gapped Graphene under a constraint that, the currents generated in response to some interaction from both the valley fermions are always equal. Interestingly, the theory in presence of this local constraint is actually an Abelian gauge theory. Ward-Takahashi identities following from this local gauge invariance are seen to yield surprising consequences. Firstly, it implies confinement of Dirac fermions, the fundamental elementary excitation of the system. Dirac fermions cease to propagate and do not show up as poles in scattering matrix. Secondly, fermion-antifermion bound pair (exciton) is seen to be allowed by the theory as a propagating mode. Contrary to belief that the theory would describe an insulator, it is found that indeed the theory shows superconductivity at low temperatures. By explicit calculations, it is seen that it possesses infinite DC conductivity, shows Meissner effect and flux quantisation. It is seen that, the Lagrange multi-

plier field introduced to implement the above constraint behaves like a Nambu-Goldstone mode of BCS theory, and plays the central role in realising superconductivity. However, unlike BCS theory, here Meissner effect and flux quantisation occur not due to Anderson-Higgs mechanism, but rather due to topological Chern-Simons coupling. After a certain finite temperature, it is observed that spontaneous proliferation of singularities in Lagrange multiplier field takes place via Berezinskii-Kosterlitz-Thouless phase transition, which marks the superconductor-insulator transition. In case of finite size Graphene with armchair egdes, the existence of dissipationless and gapless chiral edge modes is seen as a consequence of superconductivity in bulk.

It is worth mentioning that, above proposed mechanism reminds one of Schwinger model: massless QED in two space time dimensions, where photon becomes dynamically massive and is confined [8]. A natural question arises, whether the above discussed models can actually be realised in Graphene or not. Longitundinal optical phonons couple to both valley fermions in Graphene like a vector field but with a relative minus sign, and one wonders, whether they can be used to realise the first model [9]. It is shown that although the dispersion of these modes is different from that of a massless mode, still by careful manipulation they can be effectively modelled by an Abelian gauge theory. However, the Abelian theory hence obtained is massive and by explicit calculation of response functions, it is shown that it fails to serve the purpose.

Unlike above microscopic calculation, the last chapter of this thesis deals with a phenomenological description of electromagnetic response of an optical fibre. As is well known, electromagnetic response of any given material is encoded in Maxwell's theory by presence of permittivity and permeability of the given material. These response functions are functions of space-time and if the external electromagnetic field is sufficiently weak, then one can show, using model scattering calculations, that they are independent of electromagnetic field as well. However, if the fields are sufficiently intense then multiphoton processes take place and even a model scattering calculations become involved. One can still study effective dynamics by allowing response functions to be functions of fields and by choosing an appropriate ansatz for these response functions. Now that one has knowledge of response functions, the dynamics can be understood easily by solving Maxwell's equations. Since response functions now are functions of fields themselves, one finds that Maxwell's equations are no longer simple linear equations. These nonlinear equations hold all the information regarding the dynamics and hence any exact solution to these is of significant value. In most situations in real life, one does not face these nonlinearities in light-matter interactions.

However, with the advent of high power lasers and optical fibres, one can easily work in domain where these nonlinear responses are important [10]. Further in an optical fibre, the dynamics of guided modes occurs only along axial direction, and hence the problem gets considerably simplified and effectively boils down to solving a nonlinear equation of motion for electromagnetic field in 1+1 dimensions. In case when one is interested in pulse propagation, with a typical pulse width being of the order of nanoseconds, the resultant equation and its solutions called solitons have been well studied [11]. However, if one goes to even shorter time scales of around hundred femtoseconds, then the nonlinearities are too difficult to allow an exact solution to equation of motion in general. We have shown that in certain situations one can indeed solve the equation, to get a new class of exact soliton solutions of bright, dark and kink type. Further, it is shown that these solitons can be unidirectional or chiral, which means that they only propagate along one direction. It is also found that these solutions possess non-trivial phase dynamics which may provide them extra stability.

Publications

This thesis is based on following papers:

1. Vivek M. Vyas, Pankaj Patel, Prasanta K. Panigrahi, Choragudi Nagaraja Kumar, and W. Greiner, "Chirped chiral solitons in the nonlinear Schrödinger equation with self-steepening and self-frequency shift", Phys. Rev. A 78, 021803(R) (2008).

2. Vivek M. Vyas, T. Soloman Raju, and T. Shreecharan, "Classical solutions for Yang-Mills-Chern-Simons field coupled to an external source", Mod. Phys. Lett. A 26, 2357 (2011).

3. Prasanta K. Panigrahi, Vivek M. Vyas, T. Shreecharan, "A theory of non-BCS type superconductivity in gapped graphene" communicated to Euro Physics Letters (an older version of the same is arXiv:0901.1034).

4. Vivek M. Vyas, Prasanta K. Panigrahi, "A constrained theory of non-BCS type superconductivity in gapped Graphene" communicated to Annals of Physics (arXiv:1107.5521).

5. Vivek M. Vyas, Prasanta K. Panigrahi,"A gauge theory description of optical phonons in graphene" under preparation.

Contents

Notations & Conventions

In this thesis most of the time, we will be working in three dimensional Minkowskii space-time (in the natural units: $\hbar = c = 1$). Unless specified otherwise, we fix our metric to be diagonal: $\eta_{\mu\nu} = (1, -1, -1)$. Also our three-vector in this context is defined to be, $x^\mu = (x_0, \vec{x})$, so that the dual vector is given by $x_\mu = \eta_{\mu\nu} x^\nu = (x_0, -\vec{x})$, where $\vec{x} = (x, y)$. This allows us to define three derivatives as:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i} \equiv \vec{\nabla} \right),$$
$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \eta^{\mu\nu} \partial_\nu = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla} \right).$$

We will use notation $x \cdot y$ to represent the inner product, $x^\mu y_\mu$. The three momentum operator is defined to be $p^\mu = i\partial^\mu$. Fourier transform of any continuous function $f(x)$ is defined as:

$$f(x) = \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot x} F(k) \tag{1}$$
$$\text{and } F(k) = \int d^3x\, e^{ik\cdot x} f(x). \tag{2}$$

This also means that δ-function is represented in Fourier space as:

$$\delta(x) = \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot x}$$
$$(2\pi)^3 \delta(k) = \int d^3x\, e^{ik\cdot x}.$$

It is often convenient to go over to Euclidean space for doing certain calculations. This is done by defining

$$x_0^E = i\, x_0,\ p_0^E = i p_0. \tag{3}$$

This amounts to, $\eta_{\mu\nu} \to -\delta_{\mu\nu}$.

Above convention will be appropriately followed when we will work in two or four space-time dimensions *i.e.*, $x^\mu = (x_0, \vec{x})$, where $\vec{x} = (x_1, \cdots, x_n)$.

We note following identities involving Levi-Civita symbol in two and three space-time dimensions respectively.

$$\epsilon_{\mu\nu}\epsilon^{\alpha\beta} = \delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu \tag{4}$$

$$\epsilon_{\mu\nu}\epsilon^{\nu\lambda} = \delta^\lambda_\mu \tag{5}$$

$$\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2 \tag{6}$$

$$\epsilon_{\mu\nu\lambda}\epsilon^{\mu\nu\rho} = 2\delta^\rho_\lambda \tag{7}$$

$$\epsilon_{\mu\nu\rho}\epsilon^{\mu\nu\rho} = 6 \tag{8}$$

$$\epsilon_{\mu\nu\rho}\epsilon^{\mu\lambda\sigma} = \delta^\lambda_\nu\delta^\sigma_\rho - \delta^\sigma_\nu\delta^\lambda_\rho \tag{9}$$

Signum function of any real variable x, will be represented as $\mathrm{sgn}(x) = \frac{x}{|x|}$. A generalisation of Gaussian integral on function space often appears in many calculations, and it is useful to note the functional integral identity

$$\int \mathscr{D}\phi \, e^{i\left(\phi\cdot\frac{K}{2}\cdot\phi + J\cdot\phi\right)} = N \, e^{-iJ\cdot\frac{K^{-1}}{2}\cdot J}, \tag{10}$$

where ϕ and J are functions of some variable, whereas K is a non singular operator defined over the function space.

Chapter 1

Dirac fermions in Graphene

1.1 Introduction

Carbon is undoutably the most important basic material for life. Owing to its unique capacity to bind with itself and with other atoms in a variety of ways, gives rise to large array of compounds with different physical and chemical properties. It is one of the chemical elements which has been known to mankind since antiquity, in form of charcoal and soot [12]. Infact, it got its name in 1789 due to A. L. Lavoisier from the Latin word *carbo* which means 'charcoal'. Diamond (from Greek words *diaphanes*, which means 'transparent'; and *adamas* which means 'invincible') and Graphite (from Greek *graphein* which means 'to write') are its best known allotropes.

Diamonds are mainly found in ancient volcanic pipes embedded in a relatively soft, dark coloured basic rock called kimberlite. They are also found in alluvial gravels and marine terrains to which they get transported over course of time. There original mode of formation is still an area of research, but it is certainly clear that high temperature and pressure play an important role in its creation. Valence electrons of carbon atoms, in diamond, are found in sp^3 hybrid state, and hence structurally each carbon atom is tetrahedrally surrounded by four equidistant neighbouring atoms at 154.45 pm distance. It is this structure and strong C-C bonding that gives it very unusual properties like extreme hardness, very high thermal conductivity, low thermal expansion coefficient and negligible electrical conductivity.

Graphite is widely found as large crystals or flakes in metamorphosed sedimentary silicate rocks. It became popular due to commercial production of lead pencils in around 1564 in England using Cumberland Graphite. It is the most stable form of carbon at normal temperature and pressure.

Unlike diamond, graphite has a layered structure, held together by van der Waals bonds. Each layer consists of a planar hexagonal network of carbon atoms, which are in sp^2 hybrid state binded together strongly by covalent bonds (see Fig. 1.1). Such single atom thin layer of graphite is called Graphene. It is this layered stacking that gives graphite its ability to be used in pencils and in lubricants, since these layers easily slide over one another. This motivates one to think of graphene as a huge aromatic molecule, like a large conglomeration of many benzene rings. In case of benzene, it is well known that the three valence electrons in each carbon atoms exist in sp^2 hybrid state, which form a deep valence bond called σ, that is responsible for the planar hexagonal ring structure. The unused p_z orbital on each atom, forms what is called the Hückel molecular orbital (HMO) or π bond, which remains half filled because of Pauli exclusion principle. This molecular orbital is formed using certain approximations, one amongst them is called the Linear Combination of Atomic Orbitals (LCAO). Under this approximation, the molecular orbital is constructed out by a linear superposition of hydrogenic p_z orbitals, localised over each carbon atom. So as a result the electrons are not localised on any of the carbon atoms and they move freely across the ring in presence of perpendicular magnetic field to give a distinct diamagnetic signature [13]. Presence of such ring currents define *aromaticity* of an organic compound. Extrapolating the case of benzene, in graphene one expects that above HMO picture would hold, and analogue of these ring currents should exist in graphene as well. However, graphene can be thought of as an infinite periodic structure, where Bloch theorem holds and states that these individual HMOs would merge to form 'bands'. It was P. R. Wallace in 1947 who first worked out the band structure of graphene and showed that the valence band (or π bonding orbital) and conduction band (or π^* antibonding orbital) touch each other at certain points in momentum space, and as a result this material should exhibit a semimetallic behaviour, although it is a covalent solid to start with [6].

This contrasts the viewpoint held by Linus Pauling and followers, who proposed what is called the Resonating Valence Bond (RVB) model, first for benzene and then for many other systems including graphene [14]. The central philosophy of this theory is that, the molecule as a whole can be thought of as a superposed state of many independent individual structures, none of them being the true state of the system. In case of benzene, the two proposed independent structures were hexagonal carbon atom rings with alternating double and single bonds. Note that here all the carbon atoms do not have same charge distribution. This description seems to work for compounds where electron-electron repulsion is strong or in compounds where d orbitals are involved (as they

are localised away from nucleus). It has been long known that RVB model fails for benzene and other aromatic compounds. Infact it can be shown, using a very simple ground state energy calculation for a long polyene (an even sided n-gon of benzene type), that HMO gives an accurate answer compared to RVB, which overestimates ground state energy [15]. However in absence of any experimental evidence, its presence or absence in graphene was debated [16]. Untill its experimental isolation by Geim and Novoselov in 2004, monolayer graphene was thought to be a hypothetical object which is intrinsically unstable [5]. The reason for this view rests in a class of theorems which were proved independently by Hohenberg, Coleman and Merwin & Wagner; in case of quantum many body systems/field theories in lower dimensions (*i.e.,* one and two spatial dimensions) in absence of long range interactions [17, 18]. They showed by considering large class of models, that no global symmetry can be broken by vacuum, to produce an ordered state, in systems in one spatial dimension at zero temperature and in two spatial dimensions at finite temperature. The central reason for this nonexistence lies in the fact that, any broken global symmetry would be accompanied by a gapless Nambu-Goldstone boson. Being gapless, these bosons are source of infrared divergences in perturbation theory, and it was shown that in absence of long range forces, these divergences can not be cured. It implies that the broken symmetry ground state is unstable against fluctuations which tend to restore the symmetry.

Using essentially a very simple but effective technique, called mechanical exfoliation, Geim and Novolselov were able to extract graphene monolayer from graphite, and spot it on a carefully chosen SiO_2 substrate [19]. Later experiments revealed that, graphene is not structurally strictlly planar but is more like a membrane which crumples in the third dimension for more stability [9]. Infact, larger sheets of graphene are known to have ripples on its surface which are indicative of this fact. Hence, above mentioned powerful theorems do not apply to graphene in the first place, let alone their consequences.

In 1984, Semenoff wrote a landmark paper that sparked considerable interest in graphene, which was then otherwise forgotten [7]. He showed by extending the work of Wallace, that low energy electronic excitations in graphene are 'relativistic' in the sense that they effectively satisfy a two dimensional Dirac equation without a mass term. The analogue of speed of light for these quasiparticles is the Fermi velocity. It is known that Dirac electrons behave in a fundamentally different way than Schrödinger electrons (that obey nonrelativistic Schrödinger equation). The most famous and drastic difference is the existence of negative energy states for the former. This

had challenged the stability of the theory, and led Dirac to propose that all these negative energy states are filled in the ground state by (infinitely many) electrons. These particles constitute what is called the Dirac or Fermi sea, and by absorbing a certain minimum amount of energy they can be excited to positive energy states, which can be identified with usual electrons. The vacant negative energy state gets singled out in the Fermi sea, much like a bubble in water, and hence can be identified as a particle, exactly same as electron but with an opposite charge, which was later found to be positron. It may be seem a bit disturbing, since Dirac equation is an equation for a single particle wave function to start with, and it seems inevitable to postulate existence of infinitely many particles in the theory for consistency. However, Dirac equation like Klein Gordon equation, when interpreted as equation of motion for quantised fields, obviates the theory from above difficulty and brings in existence of Fermi sea in a natural way. So in a quantised Dirac field theory, there naturally exists charged fermions 'electrons' and their oppositely charged antiparticles 'positrons', which in case of graphene can be naturally identified as 'holes' in the valence band. So one sees that Semenoff's effective Dirac theory for graphene, naturally incorporates both electrons and holes in the same framework. This is in contrast to other semiconductors, where Schrödinger theory holds, and both electrons and holes obey in general different equations of motion. If this theory and interpretation correctly captures low energy electronic dynamics of graphene, then in response to external electric field there should not be any difference in current due to motion of electrons or holes, or in other words electron and hole mobilities should be identical. It is this ambipolarity that was first observed by Novoselov *et. al.* that opened gates to an era of relativistic physics in condensed matter systems [5]. Several consequences of Dirac electrons were later on experimentally verified in graphene, some of them include observation of anomalous integer quantum Hall effect and Shubnikov-de Haas oscillations [20, 21], chiral tunneling or Klein paradox [22], absence of back-scattering [23, 24], Andreev reflection [25, 26], and a direct observation of conical electronic dispersion in angle resolved photoemission spectroscopy (ARPES) experiment [27]. These firmly established the 'band' picture of graphene due to Wallace and also confirm the validity of effective Dirac theory of Semenoff.

While graphene has offered a new testing ground for ideas of high energy physics and relativistic field theories, it has also given many surprises. It is was shown by Novoselov *et. al.* that above two dimensional Dirac description is so robust in graphene that it exhibits integer quantum Hall effect even at room temperature [28]. This means that electron dynamics in graphene is unusually

stable against thermal fluctuations and disorder, and system maintains its quantum coherence. Further, it was found that the mean free path for an electron in graphene is approximately around a micrometer, which means that an electron can simply pass through the sample unimpeded implying ballistic transport [29]. At low temperature and neutral filling, that is when Fermi level exactly lies between valence and conduction bands, strangely graphene shows non zero conductivity which is independent of impurity concentration [29]. Several calculations seem to agree that conductivity is 'universal', and is given by $\frac{4e^2}{h}$, and is only determined by material independent fundamental constants. However the observed value seems to be larger than this and has a finite variance. Using an atomic force microscope (AFM), the spring constant and breaking strength of suspended graphene sheets has been measured, to find that graphene is the strongest material ever encountered [30, 31].

1.2 Tight Binding description of Graphene

As mentioned earlier, graphene is an atomically thin hexagonal structure of carbon atoms held together by covalent bonds. The valence shell of these carbon atoms consists of 4 electrons with configuration $2s^2 2p^2$. These atomic orbitals undergo sp^2 hybridisation, as a result two s and one p merge to give rise to three sp^2 hybrid orbitals which are coplanar and have trigonal symmetry. These hybrid atomic orbitals linearly combine amongst themselves to yield σ molecular orbital or the σ bond, which is responsible for strong covalent bonding, with bond length 0.142 nm. Graphene's hexagonal structure and its unusual stability owes its origin to these bonds. As noted before, one unused p_z orbital is the one that is responsible for the characteristic electronic property of this material.

Hexagonal structure itself does not form a Bravais lattice, but can be viewed as a triangular lattice structure with a unit cell consisting of two atoms (see Fig. (1.1)). Equivalently, this can also be viewed as a structure of two interpenetrating triangular sublattices. In the semiempirical nearest neighbour tight binding (or equivalently LCAO) description the Hamiltonian in second quantised form, describing the dynamics of carbon p_z orbitals or π molecular orbital is given by:

$$H_0 = -t \sum_{i \epsilon A} \sum_{a=1,2,3} \left[\hat{b}^\dagger_{i+a} \hat{a}_i + \hat{a}^\dagger_{i+a} \hat{b}_i \right]. \tag{1.1}$$

Here, $\hat{a}_i(\hat{b}_i)$ are the electron annhilation operator at site i of A(B) sublattice, and follow anticom-

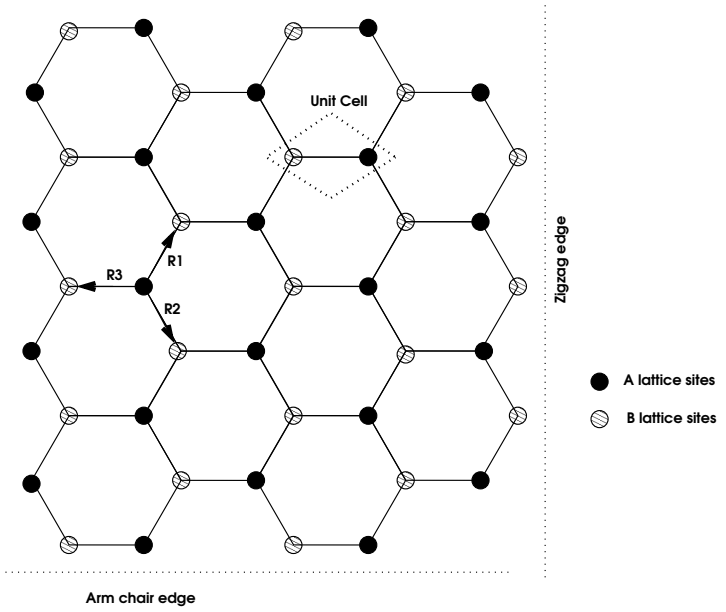


Figure 1.1: *Graphene*

mutation relations:

$$\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij},\ \{\hat{b}_i, \hat{b}_j^\dagger\} = \delta_{ij}.$$

The nearest neighbour hopping integral t is around 2.7 eV. The Bloch wavefunction defined over A(B) sublattice with wavevector $\vec{k}$ is given by:

$$|\Psi_A(\vec{k})\rangle = \frac{1}{\sqrt{N}} \sum_{i\epsilon A} e^{i\vec{k}\cdot\vec{r}_i} \hat{a}_i^\dagger |0\rangle,$$
$$|\Psi_B(\vec{k})\rangle = \frac{1}{\sqrt{N}} \sum_{i\epsilon B} e^{i\vec{k}\cdot\vec{r}_i} \hat{b}_i^\dagger |0\rangle,$$

where N is the total number of hexagonal cells. The off diagonal matrix elements of H_0 are given by:

$$\langle\Psi_A(\vec{k})|H_0|\Psi_B(\vec{k})\rangle = -t \sum_{i=1,2,3} e^{i\vec{k}\cdot\vec{R}_i},$$
$$\langle\Psi_B(\vec{k})|H_0|\Psi_A(\vec{k})\rangle = -t \sum_{i=1,2,3} e^{-i\vec{k}\cdot\vec{R}_i},$$

and the diagonal matrix elements are taken to be zero. Here, $\vec{R}_i$ are the position vectors denoting position of B lattice sites relative to A. This allows one to write the energy eigen value problem in

a matrix form:

$$E(\vec{k})\begin{pmatrix}|\Psi_A(\vec{k})\rangle\\|\Psi_B(\vec{k})\rangle\end{pmatrix} = -t\begin{pmatrix}0 & f(k)\\ f^*(k) & 0\end{pmatrix}\begin{pmatrix}|\Psi_A(\vec{k})\rangle\\|\Psi_B(\vec{k})\rangle\end{pmatrix}, \tag{1.2}$$

where, $f(\vec{k}) = \sum_{i=1,2,3} e^{i\vec{k}\cdot\vec{R}_i}$. The band structure of graphene is obtained by solving the eigenvalue problem:

$$\begin{vmatrix}E(k) & tf(k)\\ tf^*(k) & E(k)\end{vmatrix} = 0,$$

whose obvious solutions are $E(k) = \pm t|f(k)|$. Solution with negative sign corresponds to valence band, whereas positive sign corresponds to conduction band. Points in momentum space where $f(k)$ vanishes, called the Dirac points, are the points where these bands touch each other [6]. It can be shown that it occurs for six different points in k-space but only two of these points are independent, as the rest can be connected by symmetry operations (see Fig. 1.2, 1.3, 1.4). These two independent points can be chosen as $K_\pm = (\frac{2\pi}{3a}, \pm\frac{2\pi}{3\sqrt{3}a})$, ($a$ is carbon-carbon atom distance $\sim$ 0.142 nm) and commonly referred to as $K_\pm$ valleys. Since, each carbon atom contributes a single electron to π molecular orbital, it is half filled or is at neutral filling. This means that these Dirac points exactly lie on Fermi level.

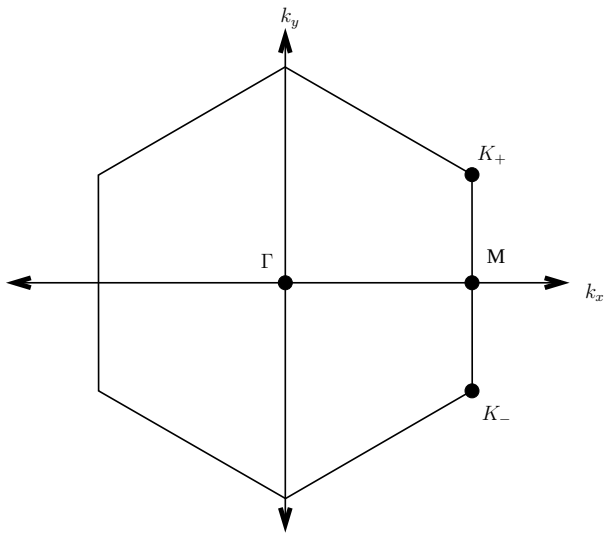


Figure 1.2: *First Brillouin zone*

As is evident, energy around these two K points varies linearly as a function of momentum.

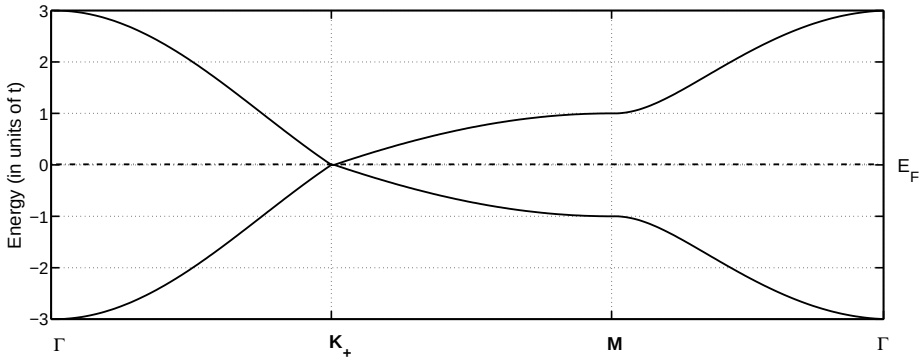


Figure 1.3: *Band structure*

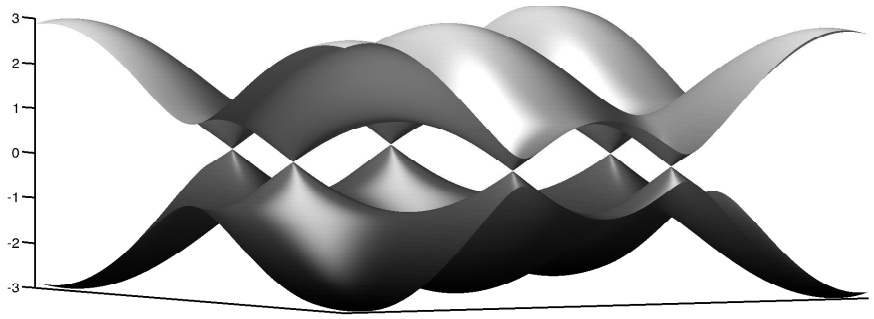


Figure 1.4: *Band structure (E as a function of $\vec{k}$)*

Infact equation(1.2) can be written around K_+ using Taylor expansion as:

$$E(\vec{k})\begin{pmatrix}|\Psi_A(\vec{k})\rangle\\|\Psi_B(\vec{k})\rangle\end{pmatrix} = \frac{3ta}{2}\begin{pmatrix}0 & k_x - ik_y\\ k_x + ik_y & 0\end{pmatrix}\begin{pmatrix}|\Psi_A(\vec{k})\rangle\\|\Psi_B(\vec{k})\rangle\end{pmatrix}.$$

This equation can be rewritten as:

$$\mathscr{H}_{K_+}|\Psi\rangle = v_F\vec{\sigma}\cdot\vec{p}|\Psi\rangle = E(p)|\Psi\rangle \tag{1.3}$$

where Fermi velocity v_F is given by $v_F = \frac{3ta}{2\hbar}$, and $\vec{\sigma} = (\sigma_x, \sigma_y)$ are Pauli matrices. Proceeding similarly, one can show that energy equation around K_- valley is given by:

$$\mathscr{H}_{K_-}|\Psi\rangle = v_F\vec{\sigma}^*\cdot\vec{p}|\Psi\rangle = E(p)|\Psi\rangle, \tag{1.4}$$

where $\vec{\sigma}^* = (\sigma_x, -\sigma_y)$. Hence, one finds that around these K points, the electronic excitations actually obey a massless Dirac equation [32, 7, 33, 34]. A natural question arises, that in the

original Dirac equation, Dirac showed that orbital angular momentum no longer commutes with the Hamiltonian, instead an additional intrinsic spin angular momentum has to be added to it, so that the net angular momentum commutes with the Hamiltonian. This intrinsic angular momentum was called the electron spin. In above case, also it is trivial to check that L_z will not commute with above Dirac Hamiltonian. One needs to add a 'spin' part S_z so that net angular momentum J_z, then commutes with the Dirac Hamiltonian. We recollect that in this framework, since the beginning, we have ignored the intrinsic spin of electron. Hence, this 'spin' can not be identified as fundamental electron spin. In the literature, this spin is referred to as pseudospin. It is natural to think whether this spin contributes to total angular momentum J_z or it is just a calculational artefact of the theory. Mecklenberg *et. al.* are of the opinion that this spin actually contributes to total angular momentum, but only when graphene is gapped [35]. In the next chapter, we will discuss this aspect in greater detail. For the moment we note that, helicity operator defined as

$$\hat{h} = \frac{1}{2}\vec{\sigma} \cdot \frac{\vec{p}}{|p|},$$

commutes with above Dirac Hamiltonian, and hence its eigen values serve as good quantum numbers.

Dirac equation around K_+ valley can be easily solved to give the eigen spinors:

$$\Psi_{\pm}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\frac{\theta(k)}{2}} \\ \pm e^{i\frac{\theta(k)}{2}} \end{pmatrix},$$

where $\pm$ sign corresponds to conduction(valence) band, and $\tan\,\theta(k) = \frac{k_y}{k_x}$. It is easy to see that above spinors are also helicity eigen states, with eigen value $\pm\frac{1}{2}$, where $+/-$ sign holds for conduction/valence band solutions. Interestingly, above solution actually exhibits spinor behaviour under rotations in k-space, since it comes back to itself not after 2π rotation but after 4π. Hence, low energy electronic excitations are actually massless Dirac fermions. It is this characteristic spinor property that is responsible for observed absence of back scattering in both carbon nanotubes and graphene, and also commonly but not incorrectly reffered to as Berry phase [24, 20]. The Dirac nature of these quasiparticles have been confirmed by several experiments, most notable being anomalous integer quantum Hall effect and Klein paradox [20, 36, 22]. One may wonder whether the occurrence of Dirac equation in graphene or boron trinitride is really unique or it can be seen elsewhere as well ? A recent paper by Corneliu shows in case of tight binding model on a network, that it possesses Dirac fermions, provided certain conditions are met [37]. Hence, the emergence of

Dirac fermions seems to be a quite general phenomena, irrespective of dimensionality of the system.

1.3 A brief review of BCS type superconductivity

Both experimentally and theoretically, possibility of superconductivity in graphene has been long sought [38, 39]. However, to the best of author's knowledge, observation of superconductivity in pure graphene has not yet been successful. Observation of proximity induced superconductivity indicates that graphene has phase coherence, and hence can support pairing [40]. Variants of usual BCS type 'pairing' mechanisms have been invoked to study possibility of superconductivity in pure and doped graphene, and graphitic layers [41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51]. The possibility of high T_c superconductivity using resonating valence bond model for doped graphene has also been studied [52, 53, 54]. It is known that, doped graphite is superconducting, and mechanisms responsible from this effect have been proposed [55, 56]. A mechanism of superconductivity based on the edge states in graphene have also been proposed [57]. A generic Hamiltonian depicting electron dynamics in cases considered above can be written as

$$H = \int d^d x \, H_K(\psi, \nabla\psi) + H_I(\psi),$$

where the first term is the kinetic part and is a function of electron field ψ and its space-time derivatives (here d is dimension of space). The second term contains the information about electron-electron interactions, which could be present effectively due to interaction with some other field say for example phonon, or simply Coulomb repulsion. In most case this term can be modeled by a four Fermi type of coupling $\psi_a^\dagger \psi_b^\dagger V_{abcd} \psi_c \psi_d$, where the precise structure of matrix V depends upon symmetry of the system and nature of interactions. In certain models, one directly works with tight binding Hamiltonian defined on the lattice, as in the previous section, rather than the continuum approximation. Here, we shall not discuss such cases, but we remark that the general philosophy in both continuum & lattice models remains same, which we shall discuss below.

Generally, the net Hamiltonian is such that under a redefinition of field ψ by a constant phase *i.e.*, $\psi \to \psi e^{i\theta}$, it retains its form. This transformation being a continuous one, gives rise to conserved charge Q, given by $\int_V \psi^\dagger \psi$, which commutes with the Hamiltonian: $[Q, H] = 0$. In presence of an external electromagnetic field, the derivative terms in kinetic part of above Hamiltonian, are replaced by covariant derivatives $D_\mu = \partial_\mu + ieA_\mu$, and Hamiltonian now remains invariant under local field transformation *i.e.,* $\psi \to \psi e^{i\theta(x)}$, which is known as gauge invariance.

Untill the path breaking work of Anderson and Nambu, amongst others including Heisenberg, it was assumed that symmetry of Hamiltonian is always realised on the ground state [2, 58, 59]. In other words, vacuum was assumed to be singlet under all the symmetry operations. However, it was shown by Anderson and Nambu that, in case of superconductivity, the U(1) gauge invariance of electrodynamics, which is realised by unitary operator $\hat{U} = e^{i\alpha Q}$, is spontaneously broken in the theory proposed by Bardeen, Cooper and Schrieffer (BCS) [60]. Equivalently, they showed that the theory realises a vacuum, that is not invariant under this symmetry operation, which means

$$\hat{Q}|0\rangle \neq 0.$$

Naively, this would simply mean that gauge invariance is lost and theory is inconsistent. However, when carefully analysed it was found that a gapless collective mode, now known as Nambu-Goldstone boson, appears in the system, which restores gauge invariance. As emphasised by Weinberg, the essential features of superconductivity can be understood only by considering existence of these Nambu-Goldstone modes [61]. Occurrence of these modes can be simply seen as a consequence of current conservation. Charge in a given theory, that couples to electric field, is defined as

$$\hat{Q} = \int d^3x \, \hat{J}_0(x),$$

where J^μ is the conserved current in the theory. Now consider an operator $\hat{\phi}(x)$ such that,

$$[\partial_\mu J^\mu, \phi(0)] = 0.$$

This implies that,

$$\int d^3x \; [\partial_\mu J^\mu, \phi(0)] = 0,$$

which can be written as

$$\int\limits_{surface} d\vec{\sigma} \cdot \Big[\nabla \cdot \vec{J}(x), \phi(0)\Big] + \partial_0 \int d^3x \; \Big[J^0(x), \phi(0)\Big] = 0.$$

In most case the surface integral vanishes and hence one can write

$$\langle 0| \, [Q, \phi(0)] \, |0\rangle = \text{const.} \neq 0.$$

Resolving identity and assuming translation invariance of the system allows one to write above

equation as

$$\langle 0|\,[Q,\phi(0)]\,|0\rangle = \sum_n (2\pi)^3\delta^3(\vec{p}_n)\,\Big[\langle 0|J_0(0)|n\rangle\langle n|\phi(0)|0\rangle e^{-iE_nt} - \langle 0|\phi(0)|n\rangle\langle n|J_0(0)|0\rangle e^{iE_nt}\Big] = \text{const.} \neq 0$$

Above equation can only hold if $E_n = 0$, which implies that there exists a gapless mode in the theory. Hence one sees that the existence of Nambu-Goldstone mode, in system where some symmetry is broken by the vacuum, is quite general. Since $\langle n|J_0(0)|0\rangle \neq 0$, these gapless modes contribute to current-current correlation $\langle [J^\mu, J^\nu]\rangle$ and hence to the electromagnetic response of the system.

In general, it can be shown that, if a theory is invariant under symmetry operations that form a Lie group denoted by G, and if the vacuum only respects the operations belonging to subgroup H of G, then there inevitably occurs gapless Nambu-Goldstone modes which live in the coset space G/H. This means that these fields, transform under G as coordinates used to label the elements of G itself, but they are identified if two of them differ by an element of H. In case of BCS theory, the gauge group $U(1) = \{e^{i\theta}|\theta \text{ is real}\}$ is spontaneously broken to $Z_2 = \{-1, 1\}$, since the vacuum is such that bilinear expectation value $\langle\psi^\dagger\psi^\dagger\rangle \neq 0$, which is usually referred to as electron-electron 'pairing'. Hence, in this case, one finds that there exists a single Nambu-Goldstone mode, denoted by $\phi(x)$ which transforms under $G = U(1)$ as phase θ *i.e.,* $\phi(x) \to \phi(x) + \Lambda(x)$, but $\phi(x)$ & $\phi(x) + \pi$ are now identified. One may write an effective action, which includes all quantum corrections, describing the interaction of such a theory with electromagnetic field as

$$L = \int d^3x\,\frac{1}{2}\left(E^2 - B^2\right) + \mathscr{L}_m[\nabla\phi - \vec{A}, \dot{f} - A_0, \chi],$$

on the grounds of gauge invariance. Here, $\mathscr{L}_m$ is a function of gauge invariant combination $\partial_\mu\phi - A_\mu$ and other gauge invariant fields χ, which represent other excitations of the theory. As argued by Anderson, the function $\mathscr{L}_m$ can not contain terms linear in $\partial_i\phi - A_i$ as long as time reversal symmetry is maintained in the system [61]. Hence, in the absence of A_0 and when ϕ is static, interaction term in the above action can be written as

$$L_m = -\frac{1}{2}\int d^3x d^3y\, C_{ij}(x-y)\left(\vec{A}(x) - \nabla\phi(x)\right)_i\left(\vec{A}(y) - \nabla\phi(y)\right)_j + \cdots,$$

where $\cdots$ represent terms with higher orders of derivatives. Positivity of energy requires that the kernel C obeys

$$\int d^3x d^3y\, C_{ij}(x-y) f_i(x) f_j(y) \geq 0,$$

for all functions f. It is the characteristic property of superconductivity that the kernel C is nonzero. Infact electromagnetic current can be found out from above action to be

$$\begin{aligned} J_i(x) &= \frac{\delta L_m}{\delta A_i(x)} \\ &= -\int d^3y\, C_{ij}(x-y)\left(\vec{A}(y) - \nabla\phi(y)\right)_j \end{aligned}$$

which is the Pippard's nonlocal formula of current density in a superconductor [1]. In absence of long range forces, the kernel C is moderately nonlocal, which roughly implies that $\vec{J} \propto \vec{A}$, which is the London equation describing response of a superconductor [62] and implies Meissner effect. Hence, one sees that the very fact that U(1) symmetry of the Hamiltonian is broken by the vacuum is sufficient to explain Meissner effect. It is shown by Weinberg that above approach also leads to prediction of all the other bulk electromagnetic responses of a superconductor like flux quantisation and Josephson current [61].

So one sees that the occurrence of Nambu-Goldstone mode due to spontaneous symmetry breaking is the sole reason for superconductivity in BCS theory. Infact above argument holds very generally, with a precise knowledge of what symmetry is being broken by vacuum, one can go ahead and construct effective action containing Nambu-Goldstone modes coupled to electromagnetic fields. Various effective Hamiltonians, in different systems as well as different models, have different symmetries and hence their symmetry breaking pattern may be very different. These inturn give rise to different Nambu-Goldstone modes and hence different effective actions. These different actions give rise to different electromagnetic responses, which finely differentiate superconductors into subclasses. Important point to note is that, irrespective of what the model is or what symmetry is broken, it is the very fact that a symmetry is broken, is the one that gives rise to superconductivity. It is this phenomenon that unifies all the existing theories of superconductivity, except class of models which are based on what are called anyons, and the models that are discussed in subsequent chapters [39].

Chapter 2

Massive particles in 2+1 dimensions

As discussed in previous chapter, the low energy electron dynamics in graphene is captured by two species of massless Dirac fermions. This means that graphene is a zero gap semiconductor. For semiconducting applications however, a finite band gap is necessary. It has been argued by Semenoff, that a small band gap can be opened up, if a local onsite onsite potential is induced such that the sublattice symmetry is broken [7]. This means that the Hamiltonian (1.1) has to be augmented by a term

$$\mathrm{H}_{onsite} = \beta \sum_i \left(a_i^\dagger a_i - b_i^\dagger b_i \right),$$

which clearly is not symmetric under exchange $a \leftrightarrow b$. Such kind of bandgap opening was reported by Zhou *et. al.* some time back [27], where it is shown that the graphene substrate interaction is the one that breaks the sublattice symmetry. It was also shown by Semenoff, that cubic Boron-Trinitride also possesses Dirac fermions as its low energy excitations, but by construction it does not have sublattice symmetry and hence their occurs a finite bandgap [7]. Presence of this finite bandgap manifests as a mass term for these Dirac fermions, without spoiling the emergent Lorentz invariance of the system. Such 'gapped' graphene has been studied by many groups since a while [35, 63, 64, 65, 66, 67, 68, 69, 70, 71]. In a Lorentz invariant theory, presence and absence of mass for particles makes significant difference in their behaviour, since massless particles are destined to travel at speed of light. Inorder to appreciate the consequences arising out of this mass term it is instructive to study the theory by considering it as a representation of Lorentz group. We have already pointed out in the previous chapter, that presence of pseudospin in graphene has led one to suspect whether it actually is a genuine angular momentum or not. In this chapter, we carefully

study the effect of presence/absence of mass on angular momentum. Also in three dimensional theories, there have been significant works on dynamical generation of mass for fermions due to interactions [72, 73, 74, 75, 76]. This possibility is interesting in its own right, however in what follows, we shall not consider such a scenario.

2.1 Poincare algebra in 2+1 dimensions

Inorder to appreciate attributes of relativistic particles with and without mass/spin in three dimensions, it is useful to study them as representations of Poincare algebra in three dimensions. To quote Wigner, 'a quantum theory describing a relativistic particle should form an irreducible representation of Poincare group'. The Lorentz group in 2+1 dimensions is SO(2,1), and consists of the three generators $L^{\mu\nu}$ which correspond to two Lorentz boosts and one rotation. The Poincare group ISO(2,1) combines the proper Lorentz group with space-time translations. The space-time translation generators P_μ alongwith $L^{\mu\nu}$ satisfy commutation relations [77, 78, 79]:

$$
\begin{aligned}
[J^\mu, J^\nu] &= i\epsilon^{\mu\nu\lambda} J_\lambda \\
[J^\mu, P^\nu] &= i\epsilon^{\mu\nu\lambda} P_\lambda \\
[P^\mu, P^\nu] &= 0
\end{aligned}
$$

where $J_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho}L^{\nu\rho}$. The irreducible single particle representations Φ of this algebra are characterised by two Casimirs: invariant mass $P^2 = P_\mu P^\mu$ and Pauli-Lubanski invariant spin $W = P_\mu J^\mu$, such that

$$
\left(P^2 - m^2\right)\Phi = 0 \text{ and } (W + ms)\,\Phi = 0,
$$

where m is defined as mass and s is spin of the particle.

Parity in three space time dimensions can not be defined as spatial reflection as in four dimensions, since in Cartesian plane vector $\vec{x}$ and $-\vec{x}$ are connected by rotation. Instead, parity is defined as

$$
(t, x, y) \to (t, -x, y)
$$

which implies that both z-component of angular momentum and Pauli-Lubanski invariant W are pseudoscalars.

2.2 Klein Gordon Theory

To understand the application of above formulation, we consider first the simplest case of Klein Gordon theory. Lagrangian for Klein Gordon theory reads:

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2,$$

for which the equation of motion is found to be

$$(\partial^2 + m^2)\phi = 0.$$

It is fairly straightforward to show that in this theory, both P^μ and J^μ, in momentum representation, are given by:

$$P^\mu\pi = p^\mu\pi \text{ and } J^\mu\pi = -i\epsilon^{\mu\nu\lambda}p_\nu\frac{\partial}{\partial p^\lambda}\pi$$

where $\pi(p)$ is Fourier transform of $\phi(x)$ [80, 81]. From these relations alongwith equation of motion, is it clear that this theory describes a massive spin zero particle.

2.3 Dirac theory

Lagrangian for massive Dirac fermions at $K_\pm$ valley is given by:

$$\mathscr{L} = \bar{\psi}_\pm \left(i\gamma^\mu_\pm\partial_\mu - m\right)\psi_\pm$$

where $\psi_\pm(x)$ is the two component spinor for $K_\pm$ valley fermions. These fermions obey the equation of motion:

$$\left(i\gamma^\mu_\pm\partial_\mu - m\right)\psi_\pm(x) = 0.$$

Here, $\gamma_\pm$ are 2×2 matrices and are defined as:

$$\gamma^0_\pm = \sigma_3,\ \gamma^1_\pm = i\sigma_1 \text{ and } \gamma^2_\pm = \pm i\sigma_2.$$

Above definition implies that, these matrices obey the identity:

$$\gamma^\mu_\pm\gamma^\nu_\pm = \eta^{\mu\nu} \mp i\epsilon^{\mu\nu\lambda}\eta_{\lambda\rho}\gamma^\rho_\pm.$$

Also note that since $i\gamma^0_\pm\gamma^1_\pm\gamma^2_\pm = \pm\mathbf{1}$, there is no counterpart of "γ_5" in three dimensional world and hence notion of chirality does not exist.

Parity operation, as defined in previous section, on spinor field is realised as:

$$\mathscr{P}\psi_{\pm}(x,y,t)\mathscr{P}^{-1} = \gamma^1\psi_{\pm}(-x,y,t),$$

where an uninteresting phase factor has been ignored. This implies that mass term $\bar{\psi}\psi$ is not invariant under this operation.

Anti unitary time reversal operation $\mathscr{T}$ implements $x_0 \to -x_0$ by changing $i \to -i$. Its action on spinor fields is found to be

$$\mathscr{T}\psi_{\pm}(x,y,t)\mathscr{T}^{-1} = \gamma^2_{\pm}\psi_{\pm}(x,y,-t).$$

Hence, mass term for Fermi fields $\bar{\psi}\psi$ violates both parity and time reversal symmetries.

Following the procedure identical to Klein-Gordon case, here it is easy to show that J^μ is given by:

$$J^\mu = -i\epsilon^{\mu\nu\rho}p_\nu\frac{\partial}{\partial p^\rho}\mathbf{1} \mp \frac{1}{2}\gamma^\mu \tag{2.1}$$

where $-(+)$ sign holds for $K_{+(-)}$ valley fermions. This alongwith equation of motion implies that this theory is describing particles with mass m and spin $s = \pm\frac{1}{2}$. This means that the spin density is given by:

$$\mathrm{S} = \pm\frac{1}{2}\psi^\dagger_{\pm}\sigma_3\psi_{\pm} = \pm\frac{1}{2}\bar{\psi}\psi.$$

Hence, one sees that mass couples to spin, and as a consequence mass term for Fermi fields is pseudoscalar. Further, one also sees that both the valley fermions carry opposite spin.

Inorder to appreciate the presence of spin in Dirac fermions, it is instructive to analyse the solutions of equation of motion:

$$\left(i\gamma^\mu_{\pm}\partial_\mu - m\right)\psi_{\pm}(x) = 0,$$

which equivalently reads in momentum space as:

$$\left(\gamma^\mu_{\pm}p_\mu - m\right)\pi_{\pm}(p) = 0,$$

where

$$\psi_{\pm}(x) = \int \frac{d^3p}{(2\pi)^3}\, e^{-ip\cdot x}\,\pi_{\pm}(p).$$

Equation of motion for K_+ valley fermions in its rest frame reads:

$$\left(\sigma_3 p_0 - m\mathbf{1}\right)\pi_+(p_0) = 0,$$

and has two obvious solutions: positive energy(particle) for which $p_0 = E$ and negative energy(hole) solutions for which $p_0 = -E$, where $E = \sqrt{\vec{p}^2 + m^2}$. In case when $m > 0$, these are:

$$\pi_+(E) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \pi_+(-E) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Interestingly, these are eigen vectors of spin operator $\mathbf{S} = \frac{1}{2}\sigma_3$ with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. This means that a free particle, in its own frame, is always pointing its spin upwards, whereas a free hole has the same pointing downwards. The same is also true when $m < 0$ except that now spin points upwards for hole and downwards for particle. Infact, it is easy to see that spin quantum number s is given by

$$s = \frac{1}{2}\text{sgn}(mp_0).$$

This is in stark contrast to four dimensional case, where there is no such restriction. A particularly interesting thing to note, is that, the charge and spin orientation in the system are correlated. Consider a scenario, when an external photon interacts with these K_+ fermions of gapped graphene to create particle-hole pair. Since, the system has charge-spin correlation, pair creation process is now allowed only for photons with a definite helicity (or spin) *i.e.,* either left or right circularly polarised photon can be absorbed depending upon whether they are coming for above or below the graphene plane. This implies that this interaction will lead to dichroism and Faraday rotation, whereby if a plane polarised light is allowed to pass through it, then plane of polarisation will be shifted.

Another way to understand this spin-charge interplay, is to look at fermion current $J_\mu = e\bar{\psi}\gamma_\mu\psi$, which couples to an external photon field A_μ. In such case the fermion current can be written using Gordon decomposition as [78]:

$$J^\mu = \frac{ie}{2m}\bar{\psi}\left(\overleftrightarrow{\partial}^\mu - 2eiA^\mu\right)\psi + \frac{e}{2m}\epsilon^{\mu\nu\lambda}\partial_\nu\left(\bar{\psi}\gamma_\lambda\psi\right),$$

where the first term represents orbital or convection part of current and the second is spin or dipole part. In case, when the system is under static external magnetic field $\vec{A}(x)$, the spin part of current which couples to it is given by:

$$J^i_{spin} = \frac{e}{2m}\epsilon^{ij}\nabla_j\rho.$$

Above relation tells us that magnetic field actually couples to charge density $\rho = \bar{\psi}\gamma^0\psi$, which means that an external magnetic field polarises charge. Infact above relation reveals that magnetic moment density is charge density.

Above analysis also holds for K_- valley fermions, which have a relative minus sign in spin operator $\mathbf{S} = -\frac{1}{2}\sigma_3$. Presence of this relative sign difference implies that these fermions behave in exactly opposite fashion when coupled to external electromagnetic fields, and as a result all the parity and time reversal violating effects, such as above mentioned dichroism, are nullified.

The fundamental difference between response of gapless and gapped graphene can now be understood by looking at massless case of above theory. Under infinitesimal rotation of angle θ, positive and negative energy solutions ($\phi^{\pm}$ respectively) of Dirac equation (for K_+ valley say) transform as:

$$\phi^{\pm}(p) \longrightarrow \left[1 \mp \frac{1}{2}i\theta + \theta\left(p_1\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial p_2}\right)\right]\phi^{\pm}(p).$$

In massless case, it is possible to redefine above solutions as:

$$\Phi^{\pm}(p) = \sqrt{p_1 \pm ip_2}\,\phi^{\pm}(p),$$

which in massive case is singular, since the prefactor vanishes in the rest frame of particle. After this redefinition it is easy to see that, although $\Phi^{\pm}$ obey massless Dirac equation and under finite rotation they transform as spinor, above infinitesimal rotation now is given by:

$$\Phi^{\pm}(p) \longrightarrow \left[1 + \theta\left(p_1\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial p_2}\right)\right]\Phi^{\pm}(p),$$

which implies that it is actually a spin zero object [77, 82]. Hence, it is often said that in three dimensions, massless fermions are spinless. This implies that the response of gapless graphene is completely different than the gapped case, as massless fermions do not carry any spin.

2.4 Abelian Gauge field

As usual, Lagrangian for Abelian gauge field is given by:

$$\mathscr{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu},$$

which by construction is gauge invariant. In functional integral formulation of quantum field theory, the vacuum functional of this theory is given by:

$$Z = \int \mathscr{D}A_\mu\, e^{i\int d^3x\, \frac{-1}{4e^2}F_{\mu\nu}F^{\mu\nu}}.$$

This can be rewritten as:

$$Z = \int \mathscr{D}A_\mu \mathscr{D}Z_{\mu\nu}\delta(F_{\mu\nu} - Z_{\mu\nu})\, e^{i\int d^3x\, \frac{-1}{4e^2}Z_{\mu\nu}Z^{\mu\nu}},$$

or equivalently

$$Z = \int \mathscr{D}A_\mu \mathscr{D}Z_{\mu\nu} \mathscr{D}\chi_\lambda \; e^{i\int d^3x \, \frac{-1}{4e^2} Z_{\mu\nu}Z^{\mu\nu} + \chi_\lambda \epsilon^{\lambda\mu\nu}(F_{\mu\nu} - Z_{\mu\nu})}.$$

Integrating A field from above expression, one finds:

$$\epsilon^{\mu\nu\lambda}\partial_\mu \chi_\nu = 0,$$

which implies that $\chi_\mu = \partial_\mu \phi$, where ϕ is a regular function. Integrating Z one obtains

$$Z = \int \mathscr{D}\phi \, e^{i\int d^3x \, \mathscr{L}},$$

where

$$\mathscr{L} = 2e^2(\partial\phi)^2.$$

This implies that the Abelian gauge theory describes bosons which are both massless and spinless. Note that this is in stark contrast to four dimensions, where the theory describes massless spin one bosons. It is tempting to identify ϕ as $\partial_\mu \phi = \epsilon_{\mu\nu\lambda}F^{\nu\lambda}$, however such an identification can lead to inconsistency. For example, in case of scalar theory, the equation of motion reads: $\partial^2\phi = 0$, which allows for singular solutions of the kind $\nabla\phi = \frac{\hat{\theta}}{r}$, for which $\nabla \times \nabla\phi = \delta(\vec{r})$. On the otherhand, in Abelian gauge theory, working in Coulomb gauge one finds that $A_0 = 0$ & $\nabla \cdot \vec{A} = 0$, which is not compatible with singularity since $\nabla \times \nabla\phi \propto \partial_0(\nabla \cdot \vec{A})$. In other words, this means that the topologically nontrivial ground state of scalar theory is not captured by Abelian gauge theory. [1]

From the works of Deser *et. al.* [83], Schonfeld [84] and Siegel [85], it became clear that in three space-time dimensions, gauge fields can be massive in a unique fashion. This mass is of topological origin, unlike mass acquired due to Anderson-Higgs mechanism, and occurs due to presence of what is called a Chern-Simons term in the action [86]. The Chern-Simons term first appeared in the work of Chern and Simons in context of differential geometry [87]. Chern Simons term for Abelian gauge field is

$$\mathscr{L} = \epsilon_{\mu\nu\lambda}A^\mu F^{\nu\lambda},$$

and clearly violates parity and time reversal. Further, the equation of motion corresponding to this action is simply $F^{\nu\lambda} = 0$. Hence, it is said that Chern-Simons term on its own possesses no

[1] For Abelian gauge theory coupled to an external current J^μ, such correspondence $\partial_\mu\phi = \epsilon_{\mu\nu\lambda}F^{\nu\lambda}$ at the level of equation of motion holds, and maps the vortices of ϕ to charges Q coupled to A_μ. This is a simple case of S-duality encountered in context of string theories.

dynamics. Note that above action is metric independent and is purely topological. Its connections with conformal field theories and knot theory have been long established [88, 89, 90]. Lagrangian for Abelian gauge field with both Maxwell and Chern-Simons term is given by:

$$\mathscr{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{M}{4}\epsilon_{\mu\nu\lambda}A^{\mu}F^{\nu\lambda},$$

with equation of motion being

$$\partial_{\mu}F^{\mu\nu} + \frac{Me^2}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0.$$

Above equation of motion can be written in terms of dual variable $\partial_{\mu}\phi = \tilde{F}_{\mu} = \epsilon_{\mu\nu\lambda}F^{\nu\lambda}$, so that it reads

$$\left(\partial^2 + M^2e^4\right)\partial_{\mu}\phi = 0,$$

which clearly implies that the theory describes particles with mass Me^2. J_{μ} for this theory is found to be [79]

$$(J^{\mu})_{\alpha\beta} = -i\epsilon^{\mu\nu\rho}p_{\nu}\frac{\partial}{\partial p^{\rho}}\delta_{\alpha\beta} + i\epsilon^{\mu}_{\alpha\beta}.$$

With this the eigenvalue equation for Pauli-Lubanski scalar reads:

$$(P\cdot J)_{\alpha\beta}\tilde{F}^{\beta} = i\epsilon_{\mu\alpha\beta}p^{\mu}\tilde{F}^{\beta} = -sm\tilde{F}_{\alpha},$$

where $s = \mathrm{sgn}(M)$ and $m = Me^2$. This clearly shows that the theory describes massive spin 1 bosons. So, as in case of fermions, here also we see that massless bosons are spinless.

It is known that in an interacting theory, where to start with one has matter fields minimally coupled to gauge fields, nonminimal coupling terms naturally arise out of quantum corrections. For example, in QED_4 to start with one has minimal coupling term $\bar{\psi}\gamma^{\mu}A_{\mu}\psi$, virtual fermion loops correct the vertex and give rise to Zeeman term $\bar{\psi}[\gamma^{\mu},\gamma^{\nu}]F_{\mu\nu}\psi$. Similarly, in case of QED_3, even when Abelian gauge field does not have a Chern-Simons term in the action to start with, fermion loops can induce it, provided fermion mass is non zero [91, 92]. Equivalently, this can be seen by calculating vacuum polarisation tensor which corrects tree level gauge field propagator. The Lagrangian governing dynamics of QED_3, is identical to QED_4 and is given by:

$$\mathscr{L}_{QED3} = \bar{\psi}\left(\gamma^{\mu}p_{\mu} - m + g\gamma^{\mu}B_{\mu}\right)\psi - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2\xi}(\partial\cdot B)^2,$$

where a covariant gauge fixing term has been added. This allows the two-point function (or propagator) for B field to be given as:

$$\langle 0|T\left(B_\alpha(x)B_\beta(y)\right)|0\rangle = iD^0_{\alpha\beta}(x-y)$$
$$D^0_{\alpha\beta}(x-y) = \int \frac{d^3k}{(2\pi)^3} e^{-ik(x-y)} \frac{(-1)}{k^2}\left[\eta_{\alpha\beta} - \frac{(1-\xi)k_\alpha k_\beta}{k^2}\right].$$

In above expression, T stands for time ordering which means $T(f(x)f(y)) = \theta(x_0-y_0)f(x)f(y) + \theta(y_0-x_0)f(y)f(x)$. In Landau gauge $\xi = 0$, which allows two-point function to have transverse form:

$$D^0_{\alpha\beta}(x-y) = \int \frac{d^3k}{(2\pi)^3} e^{-ik(x-y)} \frac{(-1)}{k^2}\left[\eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right].$$

One loop corrected propagator is given by:

$$(D^{\mu\nu}(p))^{-1} = (D^{\mu\nu}_0(p))^{-1} + \Pi^{\mu\nu}(p),$$
$$\text{where } \Pi^{\mu\nu}(p) = ig^2 \mathrm{Tr} \int \frac{d^3k}{(2\pi)^3}\gamma^\mu \frac{1}{\not{k}-m}\gamma^\nu \frac{1}{\not{k}+\not{p}-m},$$

is the vacuum polarisation tensor [34, 86, 93]. A straightforward calculation yields:

$$\Pi^{\mu\nu}(p;m) = -\frac{g^2}{\pi^2}\int_0^1 dx \left[\frac{\eta^{\mu\nu}}{3}\int_0^\infty dk + \frac{x(x-1)\pi}{2\sqrt{Q^2}}(\eta^{\mu\nu}p^2 - p^\mu p^\nu) - \frac{im\pi}{4\sqrt{Q^2}}\epsilon^{\mu\nu\lambda}p_\lambda\right], \tag{2.2}$$

where $Q^2 = x(x-1)p^2+m^2$. Above expression, as is evident, is divergent and needs to be regulated. A convenient choice is Pauli-Villars regularisation, using which regularised vacuum polarisation tensor is given by [34]:

$$\Pi^{\mu\nu}_{reg}(p) = \Pi^{\mu\nu}(p;m) + c_1\Pi^{\mu\nu}(p;\Lambda_1) + c_2\Pi^{\mu\nu}(p;\Lambda_2),$$

where Λs represent mass of heavy regulator Fermi fields. Constants $c_{1,2}$ are required to obey algebraic equation $1+c_1+c_2 = 0$. Notice that the term involving Levi-Civita symbol violates parity and is actually the Chern-Simons term. Infact it indicates that one-loop effective action governing the dynamics of gauge field, indeed has a Chern-Simons term $\epsilon^{\mu\nu\lambda}B_\mu\partial_\nu B_\lambda$, apart from kinetic term. Note that coefficient of Chern-Simons term has a discontinuity starting from two particle threshold $p^2 = 4m^2$. This implies that the intermediate fermion-antifermion state is actually responsible for occurrence of Chern-Simons term, since both carry spin $\frac{1}{2}sgn(m)$ and because angular momentum is additive in three dimensions, it adds up to give gauge boson net spin $sgn(m)$ [78]. Hence, one sees that the origin of gauge boson mass and spin lies in fermion mass/spin.

2.5 Conclusion

We saw in this chapter, that the notion of mass and spin in three dimensional world are intimately interlinked. Hence, we expect that the electromagnetic response of gapless graphene and gapped graphene would be fundamentally different. In subsequent two chapters, we will examine the consequences of the same while looking at electromagnetic response of graphene when subjected to different gauge fields.

Chapter 3

Non-BCS type superconductivity

In the previous chapter, we saw that massive fermions/bosons behave in a fundamentally different fashion then their massless counterparts. In this chapter, we show that this difference can give rise to surprisingly strange consequences. We shall construct two different models, where Dirac fermions are coupled to gauge fields, and study their electromagentic responses using two different formalisms in low energy limit.

As mentioned in previous chapter, Lagrangian (density) describing electronic excitations of gapped graphene is given by:

$$\mathscr{L}_D = \bar{\psi}_+(i\gamma_+^\mu \partial_\mu - m)\psi_+ + \bar{\psi}_-(i\gamma_-^\mu \partial_\mu - m)\psi_-,$$

where, Greek indicies μ, ν run over $0, 1, 2$. Gamma matrices for ψ_+ field are defined as $\gamma_+^0 = \sigma_3, \gamma_+^1 = i\sigma_1$ and $\gamma_+^2 = i\sigma_2$. Gamma matrices for ψ_- field are also same as ψ_+ except for γ^2, which is defined as $\gamma_+^2 = -\gamma_-^2$. As it stands, above Lagrangian is invariant under two types of independent global transformations [1]:

$$\psi_+(r) \to e^{-i\theta}\psi_+(r),\ \psi_-(r) \to e^{-i\theta}\psi_-(r), \tag{3.1}$$

$$\psi_+(r) \to e^{-i\lambda}\psi_+(r),\ \psi_-(r) \to e^{i\lambda}\psi_-(r). \tag{3.2}$$

Here θ and λ are continuous parameters, which can take values over whole of real line. Since these are continuous symmetry operations, Noether theorem holds [34], and as a result one finds two

[1] Independence of these two transformations can be easily shown by working in a reducible representation like in Ref. [94].

independently conserved currents:

$$\partial_\mu(j^\mu_+ + j^\mu_-) = 0 \text{ and } \partial_\mu(j^\mu_+ - j^\mu_-) = 0,$$

where $j^\mu(r) = \bar{\psi}(r)\gamma^\mu\psi(r)$. It is easy to see that, above relations imply conservation of both the valley currents separately, which means that no intervalley scattering takes place.

Transformations of first type (3.1) can be made local by coupling these fermions to external electromagnetic field A_μ minimally. Lagrangian describing this interaction reads:

$$\begin{aligned}\mathscr{L} =&\bar{\psi}_+(i\gamma^\mu_+\partial_\mu - m + \gamma^\mu_+ A_\mu)\psi_+ \\ &+ \bar{\psi}_-(i\gamma^\mu_-\partial_\mu - m + \gamma^\mu_- A_\mu)\psi_-,\end{aligned} \tag{3.3}$$

which remains invariant under local gauge transformations:

$$\begin{aligned}&\psi_+(r) \to e^{-i\Lambda(r)}\psi_+(r), \psi_-(r) \to e^{-i\Lambda(r)}\psi_-(r), \\ &\text{and} \quad A_\mu \to A_\mu + \partial_\mu\Lambda(r),\end{aligned} \tag{3.4}$$

where $\Lambda(r)$ is an analytic function of $(t, \vec{r})$, unlike earlier case (3.1) where it was simply a constant. Above Lagrangian captures the essentials of electromagnetic response of graphene, and is the starting point for discussions like Quantum Hall effect [20, 36].

It is natural to ask what would be electromagnetic response, if the transformation of second type becomes local. It turns out that there are two different ways by which this can occur. One is by assuming existence of an additional gauge field a_μ, which couples to $j^\mu_+ - j^\mu_-$. And the other by assuming a scenario in which some external interaction is such that it imposes a local current constraint $j^\mu_+ = j^\mu_-$. Below we shall consider both the scenarios one by one.

3.1 Model 1

Here, we shall consider effects due to Abelian gauge field a_μ, which is dynamical and which makes transformations of second type (3.2) local. Also it is assumed that it is defined on XY plane. The simplest Lagrangian, which satisfies these conditions is given by:

$$\begin{aligned}\mathscr{L} =&\bar{\psi}_+(i\gamma^\mu_+\partial_\mu - m + \gamma^\mu_+ a_\mu)\psi_+ \\ &+ \bar{\psi}_-(i\gamma^\mu_-\partial_\mu - m - \gamma^\mu_- a_\mu)\psi_- - \frac{1}{4\tilde{g}^2} f_{\mu\nu}f^{\mu\nu},\end{aligned} \tag{3.5}$$

and remains invariant under local gauge transformations:

$$\begin{aligned}&\psi_+(r) \to e^{-i\chi(r)}\psi_+(r), \psi_-(r) \to e^{i\chi(r)}\psi_-(r),\\ &\text{and} \;\; a_\mu \to a_\mu + \partial_\mu\chi(r),\end{aligned} \tag{3.6}$$

as desired. In the above Lagrangian, kinetic term for a_μ field is defined exactly like in case of electrodynamics. The field strength tensor, $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, so that 'electric field' $f_{0i} = e_i = -\dot{a}_i - \nabla_i a_0$ and 'magnetic field' $f_{12} = -b$, which allows kinetic term to be written as $\frac{1}{2}(\mathbf{e}^2 - b^2)$ [34]. In above action, $\tilde{g}$ stands for coupling constant, which unlike QED_4, has dimension of square root of mass. It can be easily shown that, the potential felt by fermions because of interaction via a_μ gauge field is logarithmic. It is known that gauge fields naturally manifest in graphene in several ways [95, 96], and so above assumption is not an unrealistic one.

3.1.1 Electromagnetic response

In functional integral formulation of quantum field theory a straightforward way to determine electromagnetic response, given a Lagrangian with electromagnetic field being external, is by defining vacuum functional for the theory [34]. Given the vacuum functional one integrates over all the intermediate fields to obtain an effective action for electromagnetic field, from which the response functions are obtained (this approach has been used in context of superconductivity in [61]). Following this approach, we start by defining vacuum functional for our theory:

$$Z = N \int \mathscr{D}[\bar{\psi}, \psi, a_\mu]\, e^{i\mathcal{S}[\bar{\psi},\psi,a_\mu,A_\mu]}, \text{ where,} \tag{3.7}$$

$$\begin{aligned}\mathcal{S} = &\int d^3x\, \bar{\psi}_+(i\gamma_+^\mu\partial_\mu - m + \gamma_+^\mu a_\mu + \gamma_+^\mu A_\mu)\psi_+\\ &+ \bar{\psi}_-(i\gamma_-^\mu\partial_\mu - m - \gamma_-^\mu a_\mu + \gamma_-^\mu A_\mu)\psi_- - \frac{1}{4\tilde{g}^2} f_{\mu\nu}f^{\mu\nu},\end{aligned} \tag{3.8}$$

where N is normalisation constant. It is worth mentioning that only three components of gauge field $A_{0,1,2}$ couple to fermions which are confined to move on the XY plane. In this analysis we shall assume for simplicity that $c = v_F = 1$, which obviously is not true in reality. However it can be shown that this does not change the qualitative features of this theory [97]. As noted earlier, fermion spectrum is gapped, and if the gap is sufficiently large then at low energy one only expects fermions as virtual excitations. So inorder to extract low energy dynamics of the system, one needs

to functionally integrate out Fermi fields in equation (3.7) to get:

$$\begin{aligned} Z =& N \int \mathscr{D}[a_\mu] \operatorname{Det}(\not{p}_+ - m + \not{\phi}_+ + \not{A}_+) \\ & \times \operatorname{Det}(\not{p}_- - m - \not{\phi}_- + \not{A}_-)\, e^{i \int d^3x \, \frac{-1}{4g^2} f_{\mu\nu} f^{\mu\nu}}. \end{aligned} \tag{3.9}$$

Above we have used Feynman slash notation, whereby $\not{x}_\pm = \gamma^\mu_\pm x_\mu$. Determinants in above expression can be easily evaluated using the method of derivative expansion [91, 92]. Above determinants can be factorized as:

$$\begin{aligned} & \operatorname{Det}(\not{p}_\pm - m \pm \not{\phi}_\pm + \not{A}_\pm) \\ & = \operatorname{Det}(\not{p}_\pm - m) \operatorname{Det}\left(1 + \frac{1}{\not{p}_\pm - m}(\pm\not{\phi}_\pm + \not{A}_\pm)\right), \end{aligned} \tag{3.10}$$

so that the first trivial determinant can be absorbed into normalization constant, and one can focus on the second non-trivial determinant known as Matthews-Salam determinant [98, 99]. This can be written as:

$$\begin{aligned} & \operatorname{Det}\left(1 + \frac{1}{\not{p}_\pm - m}(\pm\not{\phi}_\pm + \not{A}_\pm)\right) \\ & = \exp\left[\operatorname{Tr}\ln\left(1 + \frac{1}{\not{p}_\pm - m}(\pm\not{\phi}_\pm + \not{A}_\pm)\right)\right], \end{aligned} \tag{3.11}$$

where trace in above expression can be expressed in coordinate basis as:

$$\operatorname{Tr}\mathcal{O} = \operatorname{tr}\int d^3x \, \langle x|\mathcal{O}|x\rangle,$$

with tr indicating trace over internal indicies. Log function in above expression, when expanded as $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$, yields a series in powers of $a + A$. Being interested in the low energy theory, it is sufficient to only concentrate on the quadratic term, which is the first non-trivial term in above expansion, and reads:

$$\begin{aligned} & \operatorname{Det}\left(1 + \frac{1}{\not{p}_\pm - m}(\pm\not{\phi}_\pm + \not{A}_\pm)\right) \\ & \approx \exp\left[-\frac{1}{2}\operatorname{Tr}\left(\frac{1}{\not{p}_\pm - m}(\pm\not{\phi}_\pm + \not{A}_\pm)\frac{1}{\not{p}_\pm - m}(\pm\not{\phi}_\pm + \not{A}_\pm)\right)\right]. \end{aligned} \tag{3.12}$$

Using identities:

$$f(x)\, p = (p + i\partial_x) f(x) \quad \text{and} \quad \frac{1}{p} = \int_0^\infty d\alpha \, e^{-\alpha p},$$

fields in the trace term can be moved past momentum, so that the trace term is in a product form $\mathrm{Tr}(A(p)B(x))$, where A and B are functions of p and x respectively. Using completeness condition of momentum eigenstates, one finds:

$$\begin{aligned}\mathrm{Tr}(A(p)B(x)) &= \mathrm{tr}\int d^3x \int \frac{d^3p}{(2\pi)^3}\,\langle x|A(p)|p\rangle\langle p|B(x)|x\rangle \\ &= \mathrm{tr}\int \frac{d^3p}{(2\pi)^3}\,A(p)\int d^3x\,B(x).\end{aligned}$$

On tracing over internal indicies, performing the momentum space integral in above expression, and retaining the lowest order term in $\frac{1}{m}$, one gets an effective action with first order quantum corrections:

$$Z = N\int \mathscr{D}[a_\mu]\, e^{i\int d^3x\,\mathscr{L}_{eff}}, \text{ where,} \tag{3.13}$$

$$\mathscr{L}_{eff}[a,A] = -\frac{1}{4\tilde{g}^2}f_{\mu\nu}f^{\mu\nu} - \frac{m}{\pi|m|}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda + \mathcal{O}(\frac{1}{m}). \tag{3.14}$$

An additional factor of 2 has been multiplied while deriving quantum correction in above expression, inorder to take into account fermion spin. We assume that fermion mass gap is sufficiently large, so that higher order corrections in above expression are negligible. As is clear, virtual fermion loops have generated a topological mixed Chern-Simons term [83, 84, 85], due to which the two gauge fields interact. As shown before, Chern-Simons term violates both parity and time reversal symmetry, but so is not true for above written mixed Chern Simons term [97]. Note that above action is invariant under two types of gauge transformations: $a_\mu \to a_\mu + \partial_\mu\chi$ and $A_\mu \to A_\mu + \partial_\mu\Lambda$.

Inorder to arrive at an effective action for electromagnetic field alone, one needs to integrate out a_μ field in above vacuum functional. However, since a_μ is a gauge field, a naive integration leads to divergence because of gauge invariance. Using a technique due to Faddeev & Popov [34, 100], one introduces gauge fixing terms in the action. To carry out gauge fixing in self consistent manner, additional ghost fields need to be introduced. It has been known that in an Abelian theory like ours, these ghost fields get decoupled and can be ignored [101]. Hence, the Lagrangian with gauge fixing term reads:

$$\mathscr{L}_{eff}[a,A] = -\frac{1}{4\tilde{g}^2}f_{\mu\nu}f^{\mu\nu} - \frac{m}{\pi|m|}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda + \frac{1}{2\xi}(\partial_\mu a^\mu)^2, \tag{3.15}$$

where ξ is a real parameter, and it can be shown that the final result would be insensitive to its specific value. Functionally integrating a_μ field in equation (3.13) using above Lagrangian, one

obtains the following effective action for electromagnetic field:

$$\mathscr{L}_{eff}[A] = \frac{\tilde{g}^2}{2\pi^2}\left(A_\mu A^\mu - A_\mu \frac{\partial^\mu \partial^\nu}{\partial^2} A_\nu\right). \tag{3.16}$$

Above effective Lagrangian is the central result from which all the electromagnetic response functions can be easily calculated, under the purview of Kubo's linear response theory [102]. Above Lagrangian can be written in manifestly gauge invariant fashion as:

$$\mathscr{L}_{eff}[A] = -\frac{\tilde{g}^2}{4\pi^2} F_{\mu\nu} \frac{1}{\partial_\mu \partial^\mu} F^{\mu\nu}. \tag{3.17}$$

Electric current induced in the system due to presence of an external electromagnetic field, can be found as [91, 102]:

$$\langle j^\mu(x)\rangle_{ind} = \frac{\delta}{\delta A_\mu(x)} \mathcal{S}_{eff}.$$

In case, when the system is subjected to an external magnetic field, such that $A_3 = 0$ then in Lorentz gauge $\partial_\mu A^\mu = 0$, finds the celebrated London equation:

$$\langle \vec{j}(x)\rangle_{ind} = -\frac{\tilde{g}^2}{\pi^2}\vec{A}(x),$$

which was proposed to describe a superconductor [62, 1]. Further, the induced current in general reads:

$$\langle j^\mu(x)\rangle_{ind} = \frac{2\tilde{g}^2}{\pi^2 \partial^2} \partial_\alpha F^{\alpha\mu},$$

and when the magnetic field effects can be ignored, at low frequencies say, for a homogeneous and isotropic graphene sheet, one finds that:

$$\vec{j}(\omega) = \sigma(\omega)\vec{E}(\omega),$$

where frequency dependent conductivity $\sigma(w)$ goes as $\frac{1}{\omega}$. This implies that graphene possesses infinite DC conductivity, which means DC current would pass through it without any resistance.

The net action governing dynamics of electromagnetic field, which sees all four dimensions, is given by:

$$\mathscr{S} = \int d^3x dt \left[-\frac{1}{4e^2} F_{ab} F^{ab}\right] + \int\limits_{XY} d^2x dt \, \langle j^\mu\rangle_{ind} A_\mu$$

where the Latin indicies a, b take values $0, 1, 2, 3$, and the integral in the second term is understood to be over XY plane with $z = 0$. Equivalently, above action can be written as:

$$\mathscr{S} = \int d^3x dt \left[-\frac{1}{4} F_{ab} F^{ab} + \langle j^\mu\rangle_{ind} \delta(z) A_\mu\right]$$

since, charge carriers are confined to move on the XY plane. This gives rise to equation of motion

$$\partial_a F^{a\nu} = -e^2 \langle j^\nu \rangle_{ind} \delta(z), \tag{3.18}$$

for $A_{0,1,2}$ fields, whereas A_3 obeys

$$\partial_a F^{a3} = 0.$$

Working in Lorentz gauge $\partial_a A^a = 0$, it is easy to see that A_3 field gets decoupled from the rest of the fields (and currents), and obeys a free field equation

$$\partial_a \partial^a A_3 = 0.$$

In what follows, we shall consider the situation, where $A_3 = 0$ to make the analysis simple. This implies that above equation of motion for other fields, which are coupled to graphene, can be written as

$$\left(\partial_a \partial^a + \delta(z) \frac{2e^2 \tilde{g}^2}{\pi^2} \right) A_\mu = 0 \tag{3.19}$$

clearly shows that photon becomes massive while it interacts with graphene. Occurrence of δ-function in above equation may look a bit bothersome. Inorder to understand its consequences, one notes that it appears because we had assumed that the charge carriers are confined to move on XY plane which is infinitesimally thin. In reality however, graphene sheet has finite thickness, roughly of the order of size of carbon atom. So it is meaningful in above expression to work with a representation of δ-function:

$$\begin{aligned} \delta(z) &= \frac{1}{L}, \text{when} -\frac{L}{2} \leq z \leq \frac{L}{2} \\ &= 0, \text{otherwise}, \end{aligned}$$

where L is the thickness of graphene sheet. Hence, the above equation is to be solved in two regions, one being the vacuum where photon mass is absent, and the other being graphene where mass term is present, and these two solutions match at the interface. The former case is trivial and allows for propagating photons. In the latter case, the equation reads:

$$\left(\partial_a \partial^a + \frac{2e^2 \tilde{g}^2}{\pi^2 L} \right) A_\mu = 0. \tag{3.20}$$

Solving above equation with the boundary condition that fields outside graphene sheet attain a constant non-zero value, it is easy to see that the only static solutions allowed are of the type

$A_i(x) = A_i(0)e^{-\frac{x}{\lambda}}$, where $A_i(0)$ is the value of field outside the graphene sheet, and $\lambda = \sqrt{\frac{\pi^2 L}{2e^2\tilde{g}^2}}$ is the penetration depth. Hence this implies that externally applied static magnetic field can only penetrate in XY direction upto length scale λ, after which it will be screened. This is nothing but Meissner effect, a hallmark of superconductivity. Although, we had used equation of motion in terms of gauge field A_μ to arrive at this result, one can derive the same from equation (3.18) directly by writing the equation of motion only in terms of $\vec{E}$ and $\vec{B}$ fields. By simple manipulation of these equations one can easily show that in the static limit B_z field solves:

$$\left(\nabla_\perp^2 - \frac{2e^2\tilde{g}^2}{\pi^2 L}\right) B_z = 0,$$

where $\nabla_\perp^2$ is Laplacian defined on XY plane. This puts our inference of Meissner effect on stonger footing as above equation is derived in a gauge invariant manner.

Furthermore, consider a situation where a static point charge is interacting with above considered electromagnetic field. The electrostatic potential generated due to this point charge, which located at origin inside the graphene sheet, solves

$$\left(-\nabla_\perp^2 + \frac{e^2\tilde{g}^2}{\pi^2 L}\right) A_0(r) = \delta(r).$$

It is trivial to see that A_0 is given by

$$A_0(r) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + \frac{1}{\lambda^2}} = \frac{1}{2\pi} K_0(r/\lambda)$$

where $\lambda = \sqrt{\frac{\pi^2 L}{e^2\tilde{g}^2}}$ and $K_0(x)$ is hyberbolic Bessel function. Asymptotically, potential goes as $\sqrt{\frac{\pi\lambda}{2r}}e^{-\frac{r}{\lambda}}$, and it is clear that photon mass provides a characteristic length scale λ after which potential rapidly falls off. This reminds one of Debye screening that occurs in plasma. Interestingly from equation (3.20) one finds that, inside graphene sheet owing to the mass term, there exists a non-propagating solution, which although has non-zero energy, but does not propagate in space. To be explicit, it is given by $A_\mu(t) = A_\mu(0)Sin(\omega_p t)$, where $\omega_p = \sqrt{\frac{e^2\tilde{g}^2}{\pi^2 L}}$, which clearly indicates that graphene sheet as a whole sustains a collective electric field oscillation, which can be identified as plasma oscillation [103]. This identification makes sense since in case of plasma, both Debye screening and collective plasma oscillations are observed. It is not surprising to see appearance of plasma oscillations in our model, since Anderson showed that they appear, whenever photon becomes (non-topologically) massive[104]. Infact by introducing kinetic term in action for photons, which sees all four dimensions, one has introduced Coulomb repulsion between graphene electrons,

since at tree level they are the only charged species to start with. Taking into account the quantum corrections, one has found that this long range Coulomb force is screened as photon gained mass. Due to this, the system supports a collective oscillation wherein both charge carriers and electric field coherently participate. This is exactly similar to BCS superconductor, where the same occurs due to Anderson-Higgs mechanism, which is altogether different from what we observe in this case. So one can now identify quanta of this massive photon field as plasmons.

Occurrence of photon mass or presence of plasmons, implies that any propagating mode can not pass through graphene undissipated unless its frequency happens to be above plasma frequency ω_p. This in other words means that presence of this kind of superconductivity can be inferred by looking at electromagnetic reflectance/transmittance of graphene.

3.1.2 Flux Quantisation

Another interesting consequence which arises out of equation (3.13) is flux quantization. This can be easily seen [105], from the observation that transformation $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$, does not leave mixed Chern-Simons term invariant. However, the action remains invariant, assuming that fields decay sufficiently quickly as one approaches the boundary. This has interesting implications, which can be understood by considering the above theory at finite temperature. A convenient procedure to do so is imaginary time formulation [106], where one does a Wick rotation from Minkowskii space-time to Euclidean space-time: $t \rightarrow -i\tau$, where $\tau \in [0, \beta]$ is a compact variable and $\beta = \frac{1}{T}$. In this Euclidean space-time, bosonic (fermionic) fields are required to satisfy periodic (anti-periodic) boundary conditions [106]:

$$F(\vec{x}, 0) = \pm F(\vec{x}, \beta).$$

These conditions alongwith analyticity, imposes restriction on choice of gauge function $\lambda(x)$: $\lambda(\beta) = \lambda(0) + 2\pi n$. It was shown in Ref. [92], while working in imaginary time as well as real time formulation that, the above procedure of finding low energy effective action holds. Following this approach one finds vacuum functional:

$$Z^{Euclid} = N \int \mathscr{D}[a_\mu]\, e^{-S_{CS}}, \text{ where} \tag{3.21}$$

$$S_{CS} = i \int_0^\beta d\tau \int d^2x\, \frac{m}{\pi |m|} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho. \tag{3.22}$$

Variation of Chern-Simons action under these restricted gauge transformations (where $\lambda(\tau)$ only depends on τ) gives:

$$\delta S_{CS} = i\frac{2m}{|m|}n\Phi, \text{ where}$$
$$\Phi = \int d^2x\, \epsilon_{ij}\partial_i A_j,$$

is the magnetic flux. Demanding invariance of vacuum functional (3.21) gives: $\delta S_{CS} = 2\pi i N$, where N is an integer. This implies that $\Phi = N\frac{\pi|m|}{m}$. This, when appropriately scaled to SI units, reads:

$$\Phi = N\left(\frac{m}{|m|}\right)\frac{hc}{2e}.$$

Hence we find that, magnetic flux is quantized in this model, with flux unit $\frac{hc}{2e}$, akin to that of a BCS superconductor. It is easy to see that flux quantisation would naturally imply persistent current. Imagine that an external magnetic pulse triggers some current in the system, and as a result there is a net non-zero flux due to it. Now if the system offers resistance to this current then it will gradually start decaying, which change flux in a continuous manner, and hence violate flux quantisation. Hence, the only way to preserve flux quantisation is by allowing persistent current.

3.1.3 Canonical approach

In last sections, we have seen that, graphene minimally coupled to a_μ field behaves like a superconductor. Interestingly, we saw that there is no BCS type fermion pairing in this theory, which also means absence of spontaneous symmetry breaking. Inorder to be doubly sure about the same and to have a cross check, we consider above theory in canonical framework. We wish to show, that while working in a trivial vacuum, it is possible to arrive at a superconducting response as found above.

As mentioned in the previous chapter, Lagrangian for K_+ fermions is given by

$$\mathscr{L} = \bar{\psi}_+(i\gamma_+^\mu\partial_\mu - m)\psi_+,$$

where ψ_+ is two component spinor field. In this section, we will only confine ourselves to K_+ valley fermions and so we will drop the + sign for brevity. Canonically conjugate momentum for ψ field is given by

$$\pi = \frac{\partial\mathscr{L}}{\partial\dot{\psi}} = -i\psi^\dagger,$$

which allows one to construct the Hamiltonian corresponding to above Lagrangian as

$$\mathrm{H} = \int d^2x\, \psi^\dagger \left(\vec{\sigma}\cdot\vec{p} + m\sigma_z\right)\psi.$$

Inorder to quantise above theory, one postulates existence of field operators $\Psi(\vec{x},t)$ and $\Pi(\vec{x},t)$ which are defined over Fock space, and obey the canonical equal-time anticommutation relations [34]:

$$\begin{aligned}
\{\Psi_i(x),\Psi_j(y)\}_{x_0=y_0} &= 0,\\
\{\Psi_i(x),\Pi_j(y)\}_{x_0=y_0} &= -i\delta_{ij}\delta^3(\vec{x}-\vec{y}),\\
\{\Pi_i(x),\Pi_j(y)\}_{x_0=y_0} &= 0.
\end{aligned}$$

Field operator Ψ can be written in terms of fermion operators as:

$$\Psi(x) = \int \frac{d^2p}{2\pi}\sqrt{\frac{m}{E}}\left(\hat{u}(\vec{p})\psi_+(\vec{p})e^{-ip\cdot x} + \hat{v}^\dagger(\vec{p})\psi_-(\vec{p})e^{ip\cdot x}\right),$$

where $E=\sqrt{\vec{p}^2+m^2}$, $\psi_\pm(p)$ are positive/negative energy solutions of Dirac equation, and fermion operators obey following anticommutation relations:

$$\begin{aligned}
\{\hat{u}(\vec{p}_1),\hat{u}^\dagger(\vec{p}_2)\} &= \delta^2(\vec{p}_1-\vec{p}_2),\\
\{\hat{v}(\vec{p}_1),\hat{v}^\dagger(\vec{p}_2)\} &= \delta^2(\vec{p}_1-\vec{p}_2),
\end{aligned}$$

alongwith vanishing of all the other possible anticommutators amongst $\hat{u}$ and $\hat{v}$. By construction above anticommutation relations are consistent with the equal anticommutation relations for field operators. With this decomposition, Hamiltonian operator (corresponding to above Hamiltonian) can be written in normal ordered form as:

$$\hat{\mathrm{H}} = \int d^2p\, E(\vec{p})\{\hat{u}^\dagger(\vec{p})\hat{u}(\vec{p}) + \hat{v}^\dagger(\vec{p})\hat{v}(\vec{p})\}.$$

Now it can be easily seen that the following realtions relations hold

$$\begin{aligned}
[\hat{\mathrm{H}},\hat{u}(\vec{p})] &= -E(\vec{p})\hat{u}(\vec{p}), \quad [\hat{\mathrm{H}},\hat{u}^\dagger(\vec{p})] = E(\vec{p})\hat{u}^\dagger(\vec{p}),\\
[\hat{\mathrm{H}},\hat{v}(\vec{p})] &= -E(\vec{p})\hat{v}(\vec{p}), \quad [\hat{\mathrm{H}},\hat{v}^\dagger(\vec{p})] = E(\vec{p})\hat{v}^\dagger(\vec{p}).
\end{aligned}$$

Hence, if one denotes ground state or vacuum state of this theory as $|0\rangle$, and is defined as an eigenstate of Hamiltonian with zero energy *i.e.,* $\hat{\mathrm{H}}|0\rangle = 0$, then state $|p\rangle = \hat{u}^\dagger(\vec{p})|0\rangle$ represents a single particle with energy $E(\vec{p})$. Also note that these states are orthonormal $\langle p|q\rangle = \delta^2(\vec{p}-\vec{q})$.

As noted before, above written Dirac Lagrangian is invariant under a continuous phase transformation:

$$\psi(r) \to e^{-i\theta}\psi(r),\ \psi^\dagger(r) \to \psi^\dagger(r)e^{i\theta},$$

where θ is a real parameter. As per the Noether theorem, there exists a conserved current which is $j^\mu = \bar{\psi}\gamma^\mu\psi$:

$$\partial_\mu j^\mu = 0.$$

This means that the total charge $Q = \int d^2x\, j^0$ is a constant of motion and hence its value remains invariant under time evolution. This analysis, although is in the framework of classical dynamics, motivates one to think that there may exist a quantum analogue of this invariant charge. Infact, it is not difficult to see, that if one assumes above definition of charge to hold in the quantum theory as well, where $j_0(x)$ is to be understood as an operator made out of field operator Ψ, then one finds that the normal ordered $\hat{Q}$ reads

$$\hat{Q} = \int d^2p\, \{\hat{u}^\dagger(\vec{p})\hat{u}(\vec{p}) - \hat{v}^\dagger(\vec{p})\hat{v}(\vec{p})\}.$$

It is obvious that from the structure of charge operator and Hamiltonian that

$$[\hat{H}, \hat{Q}] = 0$$

and hence even in quantum theory charge operator is a constant of motion. Further, this also implies that energy eigenstates possess a well defined charge. Infact the classification of energy eigenstates becomes clear by noting following relations involving fermion operators and charge operator

$$[\hat{Q}, \hat{u}(\vec{p})] = -\hat{u}(\vec{p}), \quad [\hat{Q}, \hat{u}^\dagger(\vec{p})] = \hat{u}^\dagger(\vec{p}),$$
$$[\hat{Q}, \hat{v}(\vec{p})] = \hat{v}(\vec{p}), \quad [\hat{Q}, \hat{v}^\dagger(\vec{p})] = -\hat{v}^\dagger(\vec{p}).$$

Now provided that the vacuum state is a state devoid of any charge *i.e.,* $\hat{Q}|0\rangle = 0$, above relations imply that the single particle state $|k, 1\rangle = \hat{u}^\dagger(\vec{k})|0\rangle$ is a charge eigenstate with eigenvalue $+1$, whereas the state $|k, -1\rangle = \hat{v}^\dagger(\vec{k})|0\rangle$ is a charge eigenstate with eigenvalue -1. Physical meaning of this charge becomes clear by noting that, when above theory is minimally coupled to electromagnetic field, electric field couples to this charge, and one finds that states created by $\hat{u}^\dagger$ are actually electrons (living in conduction band) and those created by $\hat{v}^\dagger$ are holes (which live in valence band). Hence, we see that the physical picture of graphene with Dirac electrons holds when the

vacuum state is trivial *i.e.,* $\hat{Q}|0\rangle = 0$. Since, in the quantum theory, above mentioned phase transformation is generated by a unitary operator $\hat{U} = e^{-i\theta\hat{Q}}$, one also concludes that ground state respects this symmetry, and the theory is implementing this symmetry in Wigner mode. On the otherhand, if the theory realises a ground state for which $\hat{Q}|0\rangle \neq 0$, then one infers that the corresponding symmetry is spontaneously broken, and one says the that the symmetry is realised in Nambu mode [107]. It is worth mentioning that in all experimentally observed superconductors, some or the other symmetry is spontaneously broken, and one can view the classic BCS theory as a prototype explaining them. But if in a theory, like the one we have been dealing with, one postulates existence of a ground state which respects all the symmetries, then one invariably rules out spontaneous symmetry breaking. In what follows, we shall assume to be in a situation where vacuum is such that it respects all the symmetries of Hamiltonian, even when the Fermi fields are interacting with other gauge fields. We note that the time ordered two point function or the Feynman Greens function for this theory is given by:

$$\langle 0|T\left(\Psi_\alpha(x)\bar{\Psi}_\beta(y)\right)|0\rangle = iS_{F,\alpha\beta}(x-y)$$

where

$$S_{F,\alpha\beta}(x-y) = \int \frac{d^3p}{(2\pi)^3}\left(\frac{\not{p}+m}{p^2-m^2+i\epsilon}\right)_{\alpha\beta} e^{-ip\cdot(x-y)}.$$

As discussed earlier, we minimally couple this theory to an Abelian gauge field a_μ, which makes above mentioned phase transformation local. The Lagrangian for this gauge field reads:

$$\mathscr{L} = -\frac{1}{4\tilde{g}^2} f_{\mu\nu}f^{\mu\nu} + a_\mu j^\mu,$$

where j_μ stands for fermion current $\bar{\psi}\gamma^\mu\psi$. This gives rise to equation of motion for free fields:

$$\partial_\mu f^{\mu\nu} = 0.$$

As for the case of fermions, in this case also one defines canonical momentum as:

$$\Pi^\mu = \frac{\partial\mathscr{L}}{\partial\dot{a}_\mu} = -f^{0\mu},$$

and one finds that Π^0 is zero, which means that the theory is constrained. Inorder to avoid this we shall work with a gauge fixed version of above Lagrangian, which was first proposed by Fermi [108, 81],

$$\mathscr{L} = -\frac{1}{2\tilde{g}^2}(\partial_\mu a_\nu)(\partial^\mu a^\nu),$$

for which the equation of motion are given by:

$$\partial^2 a_\mu = 0.$$

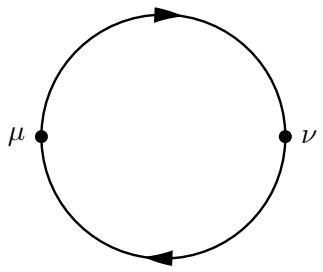
Proceeding with this Lagrangian it can be easily shown that canonical quantisation of this theory can be done without any pathologies [108, 81]. Precise calculational details of the same are not relevant for us at this juncture and we will not dwelve on it. With the assumption that the ground state of the theory is such that vacuum expectation of gauge field operator $\langle \hat{a}_\mu \rangle$ is zero, we note that the Feynman propagator is given by:

$$\langle 0|T\left(\hat{a}_\alpha(x)\hat{a}_\beta(y)\right)|0\rangle = -i\int \frac{d^3p}{(2\pi)^3}\,\frac{\eta_{\alpha\beta}}{p^2}e^{-ip\cdot(x-y)}.$$

External electromagnetic field couples to fermions minimally *i.e.,* $\bar{\psi}\gamma^\mu\psi A_\mu$, and our aim is to find out the net induced current in the system when it is subjected to such an external field. In linear response regime, the Kubo linear response formula holds [109], which is

$$J^\mu_{ind}(\vec{x},t) = -i\int d^2x' \int_{-\infty}^{\infty} dt' \,\langle [j^\mu(\vec{x},t), j^\nu(\vec{x}',t')]\rangle_{ret}\, A_\nu(\vec{x}',t'),$$

where J^μ_{ind} is the induced current, A_ν is the external electromagnetic field, whereas $\langle [j^\mu(\vec{x},t), j^\nu(\vec{x}',t')]\rangle_{ret} = i\Gamma^{\mu\nu}(x-x')$ is the retarded current-current correlation function of the theory or vacuum polarisation tensor. The leading order contribution to $\Gamma^{\mu\nu}(x-x')$ comes from the one loop 'bubble' term in perturbation series $i\Pi^{\mu\nu}(x-x')$:



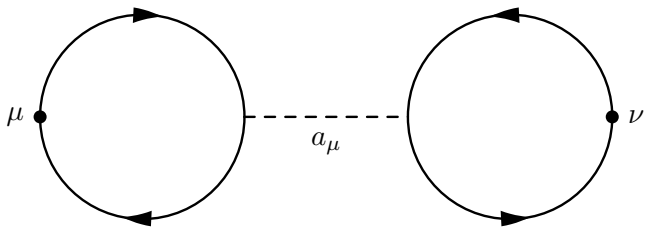
and was evaluated in the previous chapter using standard diagram technique (see equation(2.2)). As found earlier, this can be written in momentum space as

$$\Pi^{\mu\nu}(p) = \Pi_1(p)(\eta^{\mu\nu}p^2 - p^\mu p^\nu) + \Pi_2(p)\epsilon^{\mu\nu\lambda}p_\lambda, \tag{3.23}$$

where both $\Pi_{1,2}(p)$ are regular at $p^2 = 0$ after appropriate regularisation [86, 93]. To be explicit they are given by:

$$\Pi_1(p) = \frac{-1}{2\pi p^2}\left[|m| + \frac{4m^2+p^2}{4|p|}\ln\left(\frac{2|m|-|p|}{2|m|+|p|}\right)\right],$$
$$\Pi_2(p) = \frac{im}{4\pi|p|}\ln\left[\frac{2|m|+|p|}{2|m|-|p|}\right]$$

As seen earlier the contribution coming from the Chern-Simons (second) term actually cancels out as one takes into account the presence of the other valley fermion. The first term, on the otherhand survives, however it is uninteresting since it only tells about renormalisation of bare charge of fermions which indicates screening effect [110]. The next to leading order contribution to Γ comes from two loop 'eyeglass' term $i\Gamma_2^{\mu\nu}$:



which comprises of two fermion loops. This can also be evaluated using using usual diagram technique, as both vertices and propagators are known. Looking at the contribution coming due to Chern-Simons terms from both the loops one finds, in limit of small p, that:

$$\Gamma_2^{\mu\nu}(p) = -\frac{\tilde{g}^2}{4\pi^2 p^2}\left(\eta^{\mu\nu}p^2 - p^\mu p^\nu\right).$$

The contribution from coming from other valley fermion is additive, and has been considered in above expression. Also a factor of 2 due to spin degeneracy have been taken into account. Infact one can go ahead and show that, if N is the number of fermion degrees of freedom, then in the limit N is large such that $\alpha = N\tilde{g}^2$ is finite, above is the leading order contribution [111]. All the other higher order loops containing more vertices go atleast as $\frac{1}{N}$, so that in large N limit these terms do not contribute. Note that above term exhibits a pole at $p^2 = 0$, very much like what happens in Schwinger model, and hence will lead to gauge invariant photon mass [101]. Origin of this pole lies in the presence of propagator for a_μ field, which connects two fermion loops, which themselves behave regularly at $p^2 = 0$ as shown above. This implies that presence of this pole is not an arefact

of perturbative expansion and hence can not be gotten rid by any kind of resummation. In light of this term, Kubo formula implies (in Lorentz gauge alongwith $A_3 = 0$):

$$J^{\mu}_{ind}(p) \propto A^{\mu}(p),$$

which is the London equation, and hence superconductivity follows. Hence we have shown while working in canonical framework, that gapped graphene in the presence of a_μ field exhibits superconductivity, albeit without any kind of spontaneous symmetry breaking since the ground state in above calculation was assumed to be trivial. This reaffirms our earlier finding and confirms our claim about triviality of the ground state. This inturn establishes that, this mechanism of superconductivity is fundamentally different than what occurs in the usual BCS type theories.

3.2 Model 2

As noted before Lagrangian describing these massive Dirac fermions can be written in a manifest Lorentz invariant form as:

$$\mathscr{L}_D = \bar{\psi}_+(i\gamma^{\mu}_+\partial_\mu - m)\psi_+ + \bar{\psi}_-(i\gamma^{\mu}_-\partial_\mu - m)\psi_-.$$

We now consider a scenario where these quasiparticles are coupled to an interaction, which leads to preservation of valley symmetry locally. In what follows, we shall not be concerned about the detailed structure of this interaction, but make use of fact that, currents generated in response to some external stimuli, from both valleys must exactly be the same locally *i.e.*, $j^{\mu}_+(x,t) = j^{\mu}_-(x,t)$ ($j^{\mu}(x) = \bar{\psi}(x)\gamma^{\mu}\psi(x)$). Further, we shall also assume that, although the fermion fields are interacting, the interaction allows above mentioned Dirac description of quasiparticles, and the net effect of interaction amounts only to the above local current constraint (LCC).

The above condition endows the theory with a local gauge invariance. Indeed it is straightforward to check that, under following local gauge transformations:

$$\psi_+ \to e^{-i\theta(x)}\psi_+ \tag{3.24}$$

$$\psi_- \to \psi_- e^{i\theta(x)}, \tag{3.25}$$

the Lagrangian transforms as:

$$\mathscr{L}_D \to \mathscr{L}_D + (j^{\mu}_+ - j^{\mu}_-)\,\partial_\mu\theta,$$

and since $j_+^\mu - j_-^\mu = 0$, the Lagrangian remains invariant under local gauge transformation. Hence, we have shown (at the classical level) that, the theory under consideration is an Abelian gauge theory, albeit without gauge field [112].

In functional integral formulation of quantum field theory, generating functional is an object of central importance, which for the above mentioned theory reads:

$$\begin{aligned} Z &= N \int \mathscr{D}[\bar{\psi}_{(+,-)}, \psi_{(+,-)}]\, \delta\left(j_+^\mu - j_-^\mu\right) e^{iS_D}, \\ \text{where } S_D &= \int d^3x\, \mathscr{L}_D. \end{aligned} \tag{3.26}$$

Delta function in the above expression is introduced to implement LCC, and can be rewritten by introducing an additional Lagrange multiplier Bose field a_μ, such that:

$$Z[\bar{\psi}_{(+,-)}, \psi_{(+,-)}] = N \int \mathscr{D}[\bar{\psi}_{(+,-)}, \psi_{(+,-)}, a_\mu]\, e^{iS_D[\bar{\psi},\psi,a_\mu]}, \tag{3.27}$$

where

$$S_D[\bar{\psi}, \psi, a_\mu] = \int d^3x \left[\bar{\psi}_+(i\partial\!\!\!/_+ - m + a\!\!\!/_+)\psi_+ + \bar{\psi}_-(i\partial\!\!\!/_- - m - a\!\!\!/_-)\psi_-\right]. \tag{3.28}$$

Modulo an unimportant normalisation constant, (3.26) and (3.27) describe same physics, as can be checked by integrating out a_μ field to reproduce the delta function constraint. This shows that above action captures dynamics of quasiparticles in graphene subject to LCC. It is evident from the structure of the above action, that it remains invariant under local gauge transformations:

$$\psi_+ \to e^{-i\theta(x)}\psi_+,\ \psi_- \to \psi_- e^{i\theta(x)},\ a_\mu \to a_\mu + \partial_\mu\theta(x). \tag{3.29}$$

Hence, one can identify a_μ as an Abelian gauge field, which is minimally coupled to two fermions, which are oppositely charged. It is easy to see that generating functional defined above is also invariant under above transformations, and no non-trivial Jacobian appears since these transformations are assumed to be regular. As a consequence, we expect that various n-point functions in this theory would obey Ward-Takahashi identities. In our case, two-point function for K_+ valley fermions, is given by:

$$\begin{aligned} iS_F(x-y) &= \langle T\left(\psi_+(x)\bar{\psi}_+(y)\right)\rangle \\ &= N \int \mathscr{D}[\bar{\psi}_+, \psi_+, a_\mu]\ \psi_+(x)\bar{\psi}_+(y)\, e^{iS_D[\bar{\psi},\psi,a_\mu]}. \end{aligned} \tag{3.30}$$

Under field redefinition (3.29), we see that propagator satisfies a Ward-Takahashi identity,

$$S_F(x-y) = e^{-i\theta(x)} S_F(x-y) e^{i\theta(y)},$$

whose only solution is $S_F(x-y) \propto \delta(x-y)$. Above identity is very powerful, since it has allowed for an exact determination of propagator in this interacting theory. Exactly similar identity would also hold for propagator of K_- fermions. It is worth mentioning, that this model is one of the rare cases where full propagator of this theory is known without any approximation. Presence of a physically observable particle in a theory, manifests as poles of propagator in momentum space. In our case, as is clearly evident, the propagator is regular everywhere in momentum space, which implies that Dirac fermion in our theory is not a propagating mode. This is particularly surprising since we started with a free Dirac theory with a constraint condition on currents, and it appears that condition is severe enough to not allow free fermion propagation.

In the absence of Dirac fermions, it is a natural to inquire about quasiparticle excitations in this theory. Inorder to answer this question, it is instructive to study the four-point function in this theory, which is defined as:

$$\begin{aligned}&\langle T\left(\psi_+(x_1)\psi_+(x_2)\bar{\psi}_+(y_1)\bar{\psi}_+(y_2)\right)\rangle = \\ &\quad N\int \mathscr{D}[\bar{\psi}_+,\psi_+,a_\mu]\ \psi_+(x_1)\psi_+(x_2)\bar{\psi}_+(y_1)\bar{\psi}_+(y_2)\, e^{iS_D}.\end{aligned}$$

Under local gauge transformation (3.29), we obtain another Ward-Takahashi identity for four-point function:

$$\begin{aligned}&\langle T\left(\psi_+(x_1)\psi_+(x_2)\bar{\psi}_+(y_1)\bar{\psi}_+(y_2)\right)\rangle = \\ &\quad e^{-i\theta(x_1)}\, e^{-i\theta(x_2)} \langle T\left(\psi_+(x_1)\psi_+(x_2)\bar{\psi}_+(y_1)\bar{\psi}_+(y_2)\right)\rangle\ e^{i\theta(y_1)}\, e^{i\theta(y_2)}.\end{aligned}$$

Apart from a trivial non-propagating solution discussed above, assuming validity of translational invariance, above equality admits a solution of the kind:

$$\langle T\left(\psi_+(x_1)\psi_+(x_2)\bar{\psi}_+(y_1)\bar{\psi}_+(y_2)\right)\rangle \propto \delta(x_1-y_1)\,\delta(x_2-y_2)\,f(x_1-x_2),$$

where f is some function of (x_1-x_2). This means that the above identity allows for propagation of composite operator $\psi(x)\bar{\psi}(y)|_{x=y}$, which describes charge neutral excitations consisting of fermion-antifermion bound states. In this case, these can be conveniently identified with exciton excitations of graphene. Hence, we see that instead of Dirac fermion, graphene with above constraint,

admits excitons as its quasiparticle excitation. It is worth mentioning, that absence of fermions as elementary excitations and occurrence of bound states in a constrained theory like above, also appeared in a model of color confinement proposed by Rajasekaran and Srinivasan, and in related works [112, 113, 114]. Interestingly, they showed that quarks and gluons (which appeared as bound states) did not propagate and were confined, whereas mesons (color neutral bound states of quarks) were propagating excitations in their model.

3.2.1 Electromagnetic response

As done in the previous model, we wish to study the electromagnetic response of this system. Treating electromagnetic field as external we intend to study the effect on it due to quantum corrections arising from the interaction with the system. The Lagrangian below summarizes the coupling of electromagnetic field with graphene under LCC:

$$\mathscr{L} = \bar{\psi}_+(i\partial\!\!\!/_+ - m + a\!\!\!/ + A\!\!\!/)\psi_+ + \bar{\psi}_-(i\partial\!\!\!/_- - m - a\!\!\!/ + A\!\!\!/)\psi_-.$$

Inorder to take into account effects due to quantum corrections, which arise from virtual fermion loop excitation, one needs to find out the effective action by integrating out fermion field. Fermion spectrum is gapped in our theory, and at low energies fermions are only excited virtually, hence integration of fermion fields is physically meaningful. Effective action upto quadratic terms in fields, obtained using derivative expansion of fermion determinant [92, 115] reads:

$$\mathscr{L}_{eff} = -\frac{1}{6\pi|m|} f_{\mu\nu} f^{\mu\nu} - \frac{m}{\pi|m|}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho, \tag{3.31}$$

where an additional factor of 2 has been multiplied to account for spin degeneracy of fermions. It can be shown that, in the limit of large m this approximation is valid and higher order terms can be neglected. As is evident, a_μ did not have a kinetic term to start with, but fermion loops have made it dynamical, and indeed it can be identified as a genuine Abelian gauge field. Further, these two gauge fields are coupled by a mixed Chern-Simons term, which has a topological nature. Notice that above Lagrangian has the same form as obtained earlier for Model 1 (see equation (3.15)). And hence one can easily see that all the analysis done for previous model will also hold for this model as well. Inorder to understand response of the system due to influence of external electromagnetic field, we integrate out a_μ field from above Lagrangian (after appropriately gauge

fixing it), to arrive at an effective action for electromagnetic field:

$$\mathscr{L}_{eff} = \frac{3|m|}{4\pi}\left(A_\mu A^\mu - A_\mu \frac{\partial^\mu \partial^\nu}{\partial^2} A_\nu\right). \tag{3.32}$$

As is evident, interaction with a_μ field has induced mass $M = \frac{3|m|}{4\pi}$ for the physical electromagnetic field and hence superconductivity follows. Now it is clear that, this model will exhibit all the effects that were shown to hold for Model 1, since both their electromagnetic response is captured by similar effective Lagrangian. It is needless to point out that in above treatment, there is no spontaneous symmetry breaking involved.

It must be mentioned that, it is surprising in some sense to see this model showing superconductivity, since there are no on shell Dirac fermions in the theory. This implies that this exotic mechanism of superconductivity does not really care about the on shell nature of fermions. In other words, the charge is certainly not carried in both these theories by fermions. This leaves us with only one option, that the a_μ gauge field is the one that carries charge. Physically this means that as a_μ field propagates, it drags along with it a cloud of virtual fermions which gives rise to superconductivity. This interpretation raises several questions, probably the most important one is: How is this superconductivity lost ? In the next section we address this issue.

3.3 Berezinskii-Kosterlitz-Thouless phase transition

We saw in previous discussions that, the mechanism behind superconductivity in Model 1 & 2 is not based on spontaneous symmetry breaking. However, it shows all the features that a usual BCS superconductor would show. To be more precise, it can be argued that except what is called the 2 Δ mode or Higgs mode [107], there may not be any experimental signature that could distinguish between the two. The absence of spontaneous symmetry breaking rules out two things. First one is that, this superconductivity phenomenon can not be understood in terms of any local order parameter or a mean field description. Second one is that, usual Landau-Ginzburg theory of first and second order phase transition would not be relevant to capture the loss of superconductivity. Interestingly, by bypassing symmetry breaking phenomena, we have also bypassed Coleman-Mermin-Wagner-Hohenberg theorem [17, 18]. It states that in three space-time dimensions, any ordered(broken symmetry) phase can not exist at finite temperature; and in two space-time dimensions an ordered phase can not exist even at zero temperature. Clearly, our mechanism is not affected from this powerful theorem.

As noted in previous chapter, a free massless Abelian gauge theory is equivalent to a free massless scalar theory in three dimension:

$$-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \equiv 2e^2(\partial\phi)^2.$$

This also remains true even when the theory is interacting. Effective Lagrangian (3.13) can be written as

$$Z = N\int \mathscr{D}[a_\mu, Z_{\mu\nu}]\delta(f_{\mu\nu} - Z_{\mu\nu})\, e^{i\int d^3x\, \frac{-1}{4\tilde{g}^2}Z_{\mu\nu}Z^{\mu\nu} + \frac{sgn(m)}{2\pi}\epsilon^{\mu\nu\rho}Z_{\mu\nu}A_\rho},$$

which is equivalent to

$$Z = N\int \mathscr{D}[a_\mu, Z_{\mu\nu}, \chi_\lambda]\, e^{i\int d^3x\, \frac{-1}{4\tilde{g}^2}Z_{\mu\nu}Z^{\mu\nu} + \frac{sgn(m)}{2\pi}\epsilon^{\mu\nu\rho}Z_{\mu\nu}A_\rho + \chi_\rho\epsilon^{\mu\nu\rho}(f_{\mu\nu} - Z_{\mu\nu})}.$$

Integrating out a_μ field from above expression, gives rise to a constraint: $\epsilon^{\alpha\beta\partial}\partial_\alpha\chi_\beta = 0$, whose obvious solution is $\chi_\mu = \partial_\mu\phi$, for some regular function f, which is ambiguous upto addition of another regular function Λ. Integrating out Z field one gets an effective action in terms of ϕ:

$$Z = N\int \mathscr{D}\phi\, e^{i\int d^3x\, 2\tilde{g}^2\left(\partial_\mu\phi + \frac{sgn(m)}{2\pi}A_\mu\right)^2}, \tag{3.33}$$

which is gauge invariant [116, 105]. As is clear, the above effective Lagrangian is in manifest London form, and all the phenomenological properties of superconductivity would follow from here [61]. As was discussed earlier, Weinberg showed that occurrence of an electromagnetically charged field $\xi(x)$, which transforms as $\xi(x) \to \xi(x) + \Lambda(x)$ under a gauge transformation, is sufficient for the theory to exhibit superconductivity. In case of theories which exhibit any kind of fermion pairing, phenomenon of spontaneous symmetry breaking takes place, whereby there inevitably appears a massless Nambu-Goldstone field, which transforms exactly like $\xi(x) \to \xi(x) + \Lambda(x)$ under a gauge transformation [3, 117]. As was argued earlier, it is presence of this Goldstone mode that is ultimately responsible for superconductivity. One can see from above Lagrangian, that effectively in our theory, ϕ field behaves exactly like the Nambu-Goldstone mode, and hence our theory exhibits genuine superconductivity. However, it is worth noting, that in our theory, the origin of ϕ field is not due to breaking of any local symmetry by vacuum. It is rather due the fact that in three dimensions both a neutral massless scalar and massless Abelian gauge field carry same degrees of freedom and in our case we have been able to trade the latter for the former. It can be checked that, the effective action for external electromagnetic field, that is obtained by integrating over ϕ field in above expression, comes out to be the same as equation (3.16), and hence one can say that above is a dual description of the original theory.

Interestingly, above Lagrangian in the absence of A_μ field resembles that of 2D XY model in continuum limit, which is $\mathscr{L} = -(\partial\theta)^2$ [17]. In context of XY model it is known that, in general, ground state can possess singularity in form a topological vortex in θ field, which means:

$$\oint_C \vec{\nabla}\theta \cdot d\vec{s} = \pm 2\pi n, \quad (n \text{ being an integer, often called the winding number}) \tag{3.34}$$

along any closed curve C that enclosed the vortex [17]. Energy required for existence of such a vortex is $E = 2\pi \ln(L/a)$, where L is the system size and a is called the vortex core radius and is a length scale associated with the vortex. As is evident, the energy required to create a single vortex is infinite in thermodynamic limit. On the otherhand, the energy required to create a vortex-antivortex pair is finite and is given by $E = 2\pi \ln(R/a)$, where R is distance separating them. Hence, it is energetically favourable to create vortex-antivortex pairs rather an isolated vortex. In case of XY model, it was shown by Berezinskii [118] and independently by Kosterlitz and Thouless [119] that, presence of vortex excitation is forbidden energetically at zero temperature, since it does not correspond to minimum of free energy, which is given as:

$$\begin{aligned} F =& E - TS \\ =& N\,(2\pi - 2T)\,ln(L/a), \end{aligned}$$

for N vortices. As is clear, if $T > T_{BKT} = \pi$, then free energy can be minimised by having N as large as possible. On the otherhand, if $T < T_{BKT}$, then free energy is only minimised by having $N = 0$. In other words, at zero temperature, vortices and antivortices are bound together due to mutual attractive potential, and hence do not move freely. However, at temperatures above a certain critical temperature, they get unbinded. This means that at critical temperature, a transition from bound vortex state to free vortex state takes place, which is known as Berezinskii-Kosterlitz-Thouless (BKT) phase transition. This is an infinite order phase transition, wherein the ordered(low temperature) phase differs from disordered(high temperature) phase in terms of topology of field, rather than symmetry. It was shown that presence of these vortices, destroys the (quasi) long range correlation present otherwise in the system. It is also known that the ordered phase in case of XY model, does not exhibit any Long Range Order, but shows what is called a Quasi Long Range Order, whereby the correlation function exhibits a power law decay asymptotically. So as in case of XY model, in the present case also one expects a BKT phase transition to take place, after which free vortex excitations could be present. The critical temperature for this transition,

using similar argument, is given by

$$\begin{aligned} T_{BKT} &= 2\pi\tilde{g}^2, \text{for Model 1} \\ &= \frac{3\pi^2|m|}{2}, \text{for Model 2.} \end{aligned}$$

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Following Ref. [120, 121], vortex ground state is taken into account, in above action by writing $\phi \to \phi_{reg} + \phi_{vor}$, where $\phi_{reg(vor)}$ is regular(vortex) part of ϕ, so that action reads:

$$\mathscr{L}_{eff} = 2\tilde{g}^2 \left(\partial_\mu\phi_{reg} + \partial_\mu\phi_{vor} + \frac{m}{2\pi|m|}A_\mu\right)^2. \tag{3.35}$$

This approach coincides with similar treatment by Popov, where the non-trivial ground states are considered as slowly varying part of ϕ field, and excited states are viewed as rapidly varying part [122]. Inorder to study the effective low energy dynamics, one integrates out the fast component to get an effective action in terms of slow variables. Using an auxiliary field ξ_μ, above can be rewritten as:

$$\mathscr{L}_{eff} = -\frac{\xi^2}{4\tilde{g}^2} + \xi_\mu\partial^\mu\phi_{reg} + \xi_\mu\partial^\mu\phi_{vor} + \frac{\text{sgn(m)}}{2\pi}\xi_\mu A^\mu. \tag{3.36}$$

Integrating out regular part of ϕ, implies a constraint: $\partial^\mu\xi_\mu = 0$, which has an obvious solution $\xi_\mu = \epsilon_{\mu\nu\rho}\partial^\nu B^\rho$, where B_μ is some gauge field. Above action written in terms of B field is given by:

$$\begin{aligned} \mathscr{L}_{eff} =& \frac{1}{4\tilde{g}^2}B_\mu(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)B_\nu \\ &+ B_\mu\left(\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\phi_{vor} + \frac{sgn(m)}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho\right). \end{aligned} \tag{3.37}$$

Integrating out auxiliary B field, one gets an effective action describing interaction of vortex current $K^\mu = \epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\phi_{vor}$ with electromagnetic field A_μ:

$$\begin{aligned} \mathscr{L}_{eff} =& -\tilde{g}^2\left[K^\mu\frac{1}{\partial^2}K_\mu + \frac{sgn(m)}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho\frac{1}{\partial^2}K_\mu\right. \\ &\left. + \frac{sgn(m)}{2\pi}\epsilon^{\mu\nu\rho}K_\mu\frac{1}{\partial^2}\partial_\nu A_\rho + \frac{1}{4\pi^2}A_\mu\left(\frac{\partial^\mu\partial^\nu}{\partial^2} - \eta^{\mu\nu}\right)A_\nu\right]. \end{aligned} \tag{3.38}$$

Notice, that the last term in above expression, is topologically trivial and is responsible for superconductivity. In absence of vortex current, it solely describes the response of this system. First term on the otherhand, describes interaction between vortices, and it can be easily shown that interaction potential between them has a logarithmic behaviour. The other two terms, in the effective action, describe interaction of vortices with electromagnetic field. So, in order to find

contribution of these charged vortices to electromagnetic response, one requires to integrate out vortex field, to yield a net effective action for electromagnetic field. Very interestingly, integration over vortex current contributes a term of the form $A_\mu \left(\frac{\partial^\mu \partial^\nu}{\partial^2} - \eta^{\mu\nu}\right) A_\nu$ which exactly cancels with the one already present in above action. This straightforwardly means that, in presence of vortices, superconductivity is destroyed [120, 121]. So we find that loss of superconductivity in this system occurs due to production of vortices by Berezinskii-Kosterlitz-Thouless transition, and transition temperature T_{BKT} happens to be the critical temperature for superconductor-to-insulator phase transition. The high temperature 'disordered' phase is an insulator because the effective action of electromagnetic field is simply $-\frac{1}{4e_r^2} F_{\mu\nu} F^{\mu\nu}$, where e_r is renormalised charge, very much like that what is seen in dielectric media. So as expected, unlike BCS theory, here superconducting-to-insulator phase transition is not a second order phase transition, but rather is an infinite order topological phase transition. This indicates that the nature of fluctuations and critical behaviour near the critical point will be very different from that what is seen in usual BCS superconductors.

3.4 Boundary Theory

The low energy effective action describing dynamics of low energy electronic excitation, in Model 1 is given by:

$$\mathscr{L} = -\frac{1}{4\tilde{g}^2} f_{\mu\nu} f^{\mu\nu} - \frac{m}{2\pi|m|} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda - \frac{m}{2\pi|m|} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda.$$

As was observed earlier, the last term in above Lagrangian is not invariant under local gauge transformation: $a_\mu \to a_\mu + \partial_\mu \Lambda$, where Λ is some regular function of x. As a result, the change in action is given by:

$$\delta S_{CS} = -\left(\frac{sgn(m)}{4\pi}\right) \int d^3x \; \epsilon^{\mu\nu\rho} \partial_\mu \left(\Lambda \, F_{\nu\rho}\right).$$

Above volume integral can be converted to a surface integral, defined on graphene boundary (a closed compact manifold), to give an action:

$$\delta S_{CS} = -\left(\frac{sgn(m)}{4\pi}\right) \int_B d^2x \; \epsilon^{\mu\nu} \Lambda \, F_{\mu\nu}.$$

This term, as it stands, is not gauge invariant, and is defined on graphene boundary, which encloses the bulk. Gauge invariance of any given theory, is a statement that, the theory is constrained, and possesses redundant variables. We observe that, our theory to start with was gauge invariant at

classical level. One loop corrections arising out of fermion loops, generate Chern-Simons term, which exhibits gauge noninvariance. Because, our theory to start with was gauge invariant, and hence constrained, consistency demands that quantum(corrected) theory should also respect the imposed constraints, and hence should be gauge invariant. The occurrence of above gauge noninvariance, simply implies that one is only looking at one particular sector of theory, and there exists other dynamical sector, whose dynamics is such that it compensates with the one above to render the total theory gauge invariance. So we demand that there must exist a corresponding gauge theory living on the boundary, defined such that it contributes a gauge noninvariant term of exactly opposite character and hence cancels the one written above. The simplest term, living on boundary, that obeys above condition is:

$$S_B = \int\limits_B d^2x \ \left[-\frac{c}{2}(\theta\partial^2\theta) + \frac{sgn(m)}{4\pi}\theta\epsilon^{\mu\nu}F_{\mu\nu} \right],$$

where $\theta(x,t)$ is Stückelberg field, which transforms like $\theta \to \theta + \Lambda$. Note, that because of its peculiar transformation property, a quadratic mass term for θ is not gauge invariant. Hence, in a gauge theory framework like this, θ field remains massless. It can be easily checked that above action is gauge invariant only on the mass-shell. Infact the equation of motion is:

$$\partial^2\theta = \frac{sgn(m)}{4\pi c}\epsilon^{\mu\nu}F_{\mu\nu},$$

which reminds one of chiral anomaly. Above bosonic action is known to be dual to Schwinger model [101], which is made up of massless Fermi fields:

$$S_B = \int\limits_B d^2x\, \bar{\psi}(i\not{\partial} - e\not{A})\psi,$$

where $e = \frac{1}{\sqrt{4\pi c}}$. It is known in case of Schwinger model, defined on a circle, that chiral current is anomalous:

$$\partial_\mu j_5^\mu \propto \epsilon^{\mu\nu}F_{\mu\nu}.$$

Hence, in our model, the edge states are chiral & gapless.

In deriving above action, we have only considered gauge invariance with respect to transformation in a_μ field. However, analogously the same may be done for A_μ field, so that net action, describing massless surface modes, coupled to both gauge fields is given by:

$$S_B = \int\limits_B d^2x\, \bar{\psi}[i\not{\partial} - e(\not{A} + \not{a})]\psi,$$

and the anomaly equation then reads:

$$\partial_\mu j_5^\mu \propto \epsilon^{\mu\nu}(F_{\mu\nu} + f_{\mu\nu}).$$

It is worth noting that, the bulk action to start with is invariant under both P and T. However, the boundary action is not. Further, all of above discussion also holds identically for Model 2.

3.5 Conclusion

We saw that, an Abelian gauge field which couples to difference of valley fermion currents in gapped graphene, can give rise to a special type of superconductivity. In usual pairing mechanisms, which describe low temperature superconductors, an effective four Fermi interaction allows for a new superconducting ground state wherein fermions are paired, and which has a lower symmetry than the Lagrangian. Hence, in such case, the superconducting state is a new vacuum, which is unitarily inequivalent to the normal perturbative vacuum, having free fermionic excitations asymptotically [107]. Further, the superconducting vacuum is a condensate of fermion pairs, and is a macroscopic object. Certain excitation of this condensate, so called amplitude mode, have been seen experimentally, establishing unambiguously its existence. In sharp contrast to this, the current proposal for superconductivity in graphene, is not described by any kind of pairing whatsoever. As we clearly saw, it is the perturbative ground state of gapped graphene which in presence of a specific gauge field shows superconductivity. This directly implies absence of any amplitude mode, since there is no condensate in this model. This feature makes the present type of superconductivity experimentally distinguishable from the conventional ones of BCS type. It remains to be seen however, whether the gauge field assumed in above discussion is realizable in graphene or not. It is well known that, optical phonons in graphene couple to Dirac fermions as vector fields [9], albeit with different sign for both valley fermions, very much like the a_μ gauge field. However, phonon vector field is massive and hence it differs fundamentally from the one that is required to generate superconductivity in above approach. In the next chapter, we shall consider the effect of such massive vector fields on the electromagnetic response of graphene.

Chapter 4

Interaction with massive Abelian gauge field

In the last chapter, we saw that when a dynamical massless Abelian gauge field is coupled to both the valley fermions in gapped graphene, with a mutual negative sign, then the system shows superconductivity. It raises many questions. Some of them we tried to understand and answer in the last chapter. We found that, the massive nature of Dirac electrons is essential for superconductivity. However, then the question arises what happens if the Abelian gauge field in the system becomes massive ? Well, firstly it needs to be clarified what does one mean exactly with massive gauge field, since we have seen in previous chapters that there are more than one ways by which mass can be attained by gauge fields. Apart from Anderson-Higgs and Chern-Simons mass, there is yet another way in which massive gauge fields are realised, that goes by the name of Stückelberg mechanism [123, 124, 125, 126]. In this chaper, we first formulate the classical and quantum theory for massive gauge field following Stückelberg. Subsequently, we find the effect of this massive gauge field on electromagnetic response by looking at low energy effective action and show that it does not lead to superconductivity.

One may wonder that, in previous chapter and in this chapter as well, we are considering graphene fermions interacting with gauge fields, but where from these fields arise ? One of the very interesting ways in which gauge fields appear in quantum systems is via Berry phase [127]. Often in atomic and molecular physics as well as in condensed matter physics, one deals with systems which are made of two subsystems whose dynamics occur at different energy/time scales. Hence, while

working at low energy, the subsystem whose energy eigenstates are widely separated (or 'fast' mode of the system) will be unaffected or in other words it will be in its stationary state throughout. Hence, the response of the system will only be governed by the subsystem whose energy eigenstates are closely spaced (or 'slow' mode of the system). If the difference of energy/time scale between the fast or slow modes is sufficiently large, then one can write the wavefunction of the total system as a product of wavefunctions of fast mode (which parameterically depends on coordinates of slow mode) and slow mode. This approximation is the famous adiabatic Born-Oppenheimer approximation. However, in many systems this approximation breaks down due to various reasons. If the energy scales of fast and slow modes are still very different, then one can integrate or average out the fast modes and get an effective Hamiltonian for slow modes. In certain systems, it turns out that the effect of fast modes manifests as an effective vector potential for slow modes, which is now called the Berry potential [127]. So one expects to see gauge fields appearing in systems where such adiabatic approximation does not hold.

While constructing the band theory of graphene, we have implicitly assumed that the hexagonal network of σ-bonds is rigid and static. In reality, however it is not the case and there exist lattice vibrations or phonons which propagate through the system and deform the lattice. Further, defects and disorder of various types are inevitable in any condensed matter system. As far as phonons are concerned, their energy scale is much smaller compared to that of electrons, which means that time for phonon passage through a unit cell is much larger when compared to a typical electronic transition time scale. This means that while two electrons can interact by exchanging a phonon, phonon dynamics can not get affected by electrons, in particular one need not consider phonon renormalisation due to electrons. This is the essence of famous Migdal theorem, which is a consequence of adiabatic Born-Oppenheimer approximation. Now in graphene if one finds that phonons actually get renormalised due to electron-hole loops then it implies that adiabatic Born-Oppenheimer approximation is no longer valid. Using Raman spectroscopy, Pisana *et. al.* looked at optical phonons in graphene which give the G-peak in the spectrum, and showed that by changing position of Fermi level from valence to conduction band, the G-peak shifts [128]. This indicated that these phonons, which occur due to vibrations in σ-bonds, do get renormalised due to electrons from π-bonds, which invalidates adiabatic Born-Oppenheimer approximation in this material. This is exactly what happens in QED for example, where both gauge and matter fields get renormalised due to quantum corrections. When phonons pass through the lattice, they change

the bond length/angle locally, which gets reflected in the hopping amplitude. This means that the tight binding Hamiltonian described in first chapter, should be generalised as:

$$H = -\sum_{i\epsilon A}\sum_{a=1,2,3}\left[t_{i,a}\hat{b}^{\dagger}_{i+a}\hat{a}_i + t^{*}_{i,a}\hat{a}^{\dagger}_{i+a}\hat{b}_i\right], \tag{4.1}$$

to accommodate this fact. If the change in hopping amplitude from the equilibrium value is small then one can write $t_{i,a} \sim t + \delta t_{i,a}$ in above Hamiltonian, and if the variations in $\delta t_{i,a}$ are much larger compared to lattice spacing then, the low energy description of graphene is still in term of $K_{\pm}$ valley fermions. The hopping amplitude $\delta t_{i,a}$ varies as distance between the two neighbouring atoms, and is altered as phonons pass by. Using these facts, Ishikawa and Ando showed, that when one expands above Hamiltonian around $K_{\pm}$ points, for in-plane transverse and longitudinal optical phonons, one finds that effective Hamiltonian is:

$$H = \psi^{\dagger}_{+}\left(v_F\vec{\sigma}\cdot(\vec{p}+\vec{a})\right)\psi_{+} + \psi^{\dagger}_{-}\left(v_F\vec{\sigma}^{*}\cdot(\vec{p}-\vec{a})\right)\psi_{-}, \tag{4.2}$$

which is nothing but Dirac Hamiltonian minimally coupled to a vector field [129]. The vector field $\vec{a}$ is connected to lattice displacement as:

$$(a_x(\vec{r}), a_y(\vec{r})) = \frac{g_{on}}{v_F l}\left(u_y(\vec{r}), -u_x(\vec{r})\right), \tag{4.3}$$

where $\vec{u}(\vec{r}) = \vec{u}_B(\vec{r}) - \vec{u}_A(\vec{r})$, and $\vec{u}_{A,B}(\vec{r})$ represent the displacement vector of (A,B) lattice point located at $\vec{r}$. The C-C bond length is given by l, whereas g_{on} is a constant arising from gradient of interatomic potential and is estimated to be 6.39 eV [129, 130]. So optical phonons couple to Dirac fermions in graphene as vector fields, unlike other materials where they usually couple to electrons as scalar fields. It is worth mentioning, that these phonons couple to both valley fermions but with a mutual sign difference. This is because vector field itself breaks parity and time reversal symmetry; the only way a system with these symmetries can accomodate vector fields is by having opposite sign coupling to both the valley fermions. It may also be noted that, in-plane acoustic phonons and strain also couple to valley fermions like the mentioned vector field [9]. Since the dispersion of acoustic phonons is $w \propto k$, and so a suitable kinetic energy term in Lagrangian governing dynamics of corresponding vector field will be

$$\mathscr{L} = \frac{1}{2}a_i\partial^2 a^i,$$

for which the equation of motion is given by:

$$\partial^2 a_i = 0.$$

Modulo absence of a_0, above action is exactly the same as the one proposed by Fermi for photon field, and that was encountered in the previous chapter. So, one can view this theory as a gauge fixed version of Model 1, albeit without any a_0. Now, if one wishes to study optical modes, which have dispersion $w^2 = k^2 + w_0^2$, then above theory needs to be generalised. This is the programme pursued in this chapter.

4.1 Massive Vector field

In the previous chapter, we had considered a situation when fermions in graphene were interacting with a massless Abelian gauge field. Consider the scenario if that gauge field is replaced by a Proca field. Lagrangian governing dynamics in such case is given by [34]:

$$\mathcal{L} = \bar{\psi}_+(\gamma_+^\mu p_\mu - m + g\gamma_+^\mu a_\mu)\psi_+ + \bar{\psi}_-(\gamma_-^\mu p_\mu - m - g\gamma_-^\mu a_\mu)\psi_- - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{\mu^2}{2}a_\mu a^\mu,$$

where g is identified as coupling constant, and has dimension square root of mass. In order to study the dynamics in detail, let us consider only K_+ species of electrons interacting with the Proca field minimally:

$$\mathcal{L} = \bar{\psi}_+(\gamma_+^\mu p_\mu - m + g\gamma_+^\mu a_\mu)\psi_+ - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{\mu^2}{2}a_\mu a^\mu. \tag{4.4}$$

The theory with K_- species is identical except $g \to -g$ & $\gamma_+^2 \to -\gamma_-^2$, so one can readily infer about K_- species once theory with K_+ species is well understood. We note that in above Lagrangian, fermion current $\bar{\psi}_+\gamma_+^\mu\psi_+$ is conserved, since it also happens to be electromagnetic charge current. The Proca field as is obvious, is not a gauge field. However, it can be cast into a gauge theory using an idea due to Stückelberg [123, 124, 125]. He noticed that, any vector field can be written as a sum of two fields:

$$a_\mu = B_\mu - \frac{1}{\mu}\partial_\mu C,$$

where B_μ & C represent the vector and scalar fields respectively. In terms of these fields Proca Lagrangian can be written as:

$$\mathcal{L}_{Proca} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{\mu^2}{2}\left(B_\mu - \frac{1}{\mu}\partial_\mu C\right)^2,$$

where $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and we have assumed that C field is nonsingular *i.e.*,$\partial_\mu\partial_\nu C = \partial_\nu\partial_\mu C$. In general, C field can have vortex singularities and in that case partial derivatives may not commute

when acted over C. However, like in 2D XY model, it can be argued that at sufficiently low temperature, vortex-antivortex pairs are bound allowing C field to be regular. However, at some finite temperature vortex-antivortex unbinding may take place, signaling occurrence of an infinite order Berezinskii-Kosterlitz-Thouless phase transition, after which C field would have singularities. In this discussion, we are confining ourselves to zero temperature scenario, assumption that C field is regular everywhere is hence justified. In this form, the Proca Lagrangian is seen to have local gauge invariance under following transformation rule:

$$\begin{aligned} B_\mu &\to B_\mu + \partial_\mu \Lambda, \\ C &\to C + \mu\Lambda, \end{aligned}$$

where Λ is arbitrary smooth function. Hence, we have been able to show that in this case too electrons are actually coupled to gauge fields, albeit gauge invariance is manifested in Stückelberg form. Equation (4.4) can be written as:

$$\begin{aligned} \mathcal{L} = \bar{\psi}_+ \left(\gamma_+^\mu p_\mu - m + g\gamma_+^\mu B_\mu - g\gamma_+^\mu \frac{1}{\mu}\partial_\mu C \right) \psi_+ \\ -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{\mu^2}{2}\left(B_\mu - \frac{1}{\mu}\partial_\mu C \right)^2 . \end{aligned}$$

Conservation of fermion current $\bar{\psi}_+ \gamma_+^\mu \psi_+$ along with the assumption that the fields consider here decay sufficiently quickly as one goes to graphene boundary, implies that the C field decouples from the fermions:

$$\mathcal{L}_{el-ph} = \bar{\psi}_+ \left(\gamma_+^\mu p_\mu - m + g\gamma_+^\mu B_\mu \right) \psi_+ - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{\mu^2}{2}\left(B_\mu - \frac{1}{\mu}\partial_\mu C \right)^2 . \tag{4.5}$$

Behaviour of Fermi fields under a gauge transformation can be better understood by defining covariant derivative $D_\mu = \partial_\mu - igB_\mu$. It can be easily seen that under a gauge transformation, with Fermi fields transforming as:

$$\psi' = \psi e^{ig\Lambda}, \quad \bar{\psi}' = \bar{\psi} e^{-ig\Lambda},$$

Covariant derivative transforms as:

$$D'_\mu \psi' = (\partial_\mu - igB'_\mu)\psi' = e^{ig\Lambda}(\partial_\mu - igB_\mu)\psi = e^{ig\Lambda} D_\mu \psi,$$

and hence fermion Lagrangian:

$$\begin{aligned} \mathcal{L}_{el} &= \bar{\psi}_+ \left(\gamma_+^\mu p_\mu - m + g\gamma_+^\mu B_\mu \right) \psi_+ \\ &= \bar{\psi}_+ \left(\gamma_+^\mu i D_\mu - m \right) \psi_+ \end{aligned} \tag{4.6}$$

remains invariant under gauge transformation.

Thus, we have shown that the total Lagrangian is invariant under gauge transformations. It is easy to see that, set of all possible gauge transformations on ψ, say $e^{ig\Lambda}$ for all possible values of Λ, forms a group. Further, since $e^{ig\Lambda_1}e^{ig\Lambda_2} = e^{ig(\Lambda_1+\Lambda_2)} = e^{ig\Lambda_2}e^{ig\Lambda_1}$, this group is Abelian U(1) with one parameter Λ.

In Lagrangian formulation of quantum field theory, the key object is the vacuum persistence amplitude or vacuum functional Z [34], which is given by a functional integral:

$$Z[j,\phi,\bar{\eta},\eta] = N\int \mathcal{D}[\bar{\psi},\psi,B_\mu,C]\, e^{iS}, \tag{4.7}$$

$$S = \int d^3x\ \mathcal{L}_{el-ph} + \phi C + j_\mu B^\mu + \bar{\eta}\psi + \bar{\psi}\eta, \tag{4.8}$$

where N stands for normalisation, $\mathcal{D}[\bar{\psi},\psi,B_\mu,C]$ represents measure for functional integral over fields $\bar{\psi}$, ψ, B_μ and C. Here ϕ, j, $\bar{\eta}$ & η represent sources to respective fields, which are usually added in this formalism to facilitate calculations involving various n-point functions, and they are set to zero at end of calculation. In case of gauge theories, because of gauge invariance, the functional integration defined as above diverges [34]. To cure this, a technique originally due to Fadeev & Popov [100] is used, and a particular gauge is chosen by introducing a gauge fixing term. This changes the original theory, and ghost fields are introduced for maintaining consistency. However, in an Abelian theory like ours, these ghost fields are known to get decoupled completely, and hence one need not consider them here [101]. Lagrangian with a linear covariant gauge fixing term is given by:

$$\mathcal{L}_{el-ph} = \bar{\psi}_+\left(\gamma^\mu_+ p_\mu - m + g\gamma^\mu_+ B_\mu\right)\psi_+ - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{\mu^2}{2}\left(B_\mu - \frac{1}{\mu}\partial_\mu C\right)^2 - \frac{1}{2\xi}(\partial\cdot B + \xi\mu C)^2,$$

where ξ is the gauge parameter. After a little algebra one finds:

$$\begin{aligned}\mathcal{L}_{el-ph} =& \bar{\psi}_+\left(\gamma^\mu_+ p_\mu - m + g\gamma^\mu_+ B_\mu\right)\psi_+ \\ &- \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{\mu^2}{2}B_\mu B^\mu - \frac{1}{2\xi}(\partial\cdot B)^2 + \frac{1}{2}(\partial C)^2 - \frac{\xi^2\mu^2}{2}C^2.\end{aligned} \tag{4.9}$$

Notice that the C field completely gets decoupled from the rest of the fields, which allows us to functionally integrate it in expression (4.7) to give an overall constant, which can be absorbed in normalisation without affecting other terms. This allows above Lagrangian to be written as:

$$\mathcal{L}_{el-ph} = \bar{\psi}_+\left(\gamma^\mu_+ p_\mu - m + g\gamma^\mu_+ B_\mu\right)\psi_+ \tag{4.10}$$

$$- \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{\mu^2}{2}B_\mu B^\mu - \frac{1}{2\xi}(\partial\cdot B)^2. \tag{4.11}$$

Thus, we have arrived at a theory of electrons interacting with a gauge field, which is an Abelian gauge theory like QED, albeit here the gauge boson is massive. This can be seen by looking at the two-point function(or propagator) for B field:

$$\langle 0|T\left(B_\alpha(x)B_\beta(y)\right)|0\rangle = \frac{i^{-2}}{Z}\frac{\delta^2 Z}{\delta j^\alpha(x)\delta j^\beta(y)}\bigg|_{j=0} = iD^0_{\alpha\beta}(x-y)$$
$$D^0_{\alpha\beta}(x-y) = \int \frac{d^3k}{(2\pi)^3}e^{-ik(x-y)}\frac{(-1)}{k^2-\mu^2}\left[\eta_{\alpha\beta} - \frac{(1-\xi)k_\alpha k_\beta}{k^2-\xi\mu^2}\right],$$

which has a pole for $k^2 = k_\mu k^\mu = w^2 - \vec{k}\cdot\vec{k} = \mu^2$. In above expression, T stands for time ordering which means $T(f(x)f(y)) = \theta(x_0-y_0)f(x)f(y) + \theta(y_0-x_0)f(y)f(x)$. In Landau gauge $\xi = 0$, which allows two-point function to have transverse form:

$$D^0_{\alpha\beta}(x-y) = \int \frac{d^3k}{(2\pi)^3}e^{-ik(x-y)}\frac{(-1)}{k^2-\mu^2}\left[\eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right].$$

The force mediated due to exchange of gauge particles between two electrons can be easily calculated from propagator in Feynman-Fermi gauge by setting $\xi = 1$, and is given by:

$$V(r) = \int \frac{d^2k}{(2\pi)^2}\frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+\mu^2} = \frac{1}{2\pi}K_0(\mu r)$$

where $K_0(x)$ is hyberbolic Bessel function. Asymptotically, potential goes as $V(r) = \sqrt{\frac{\pi}{2\mu r}}e^{-\mu r}$, and we see that mass μ provides a characteristic length scale after which potential rapidly falls off. It is worth mentioning, that the potential is analytic at $r = \infty$, unlike massless theory where potential typically goes as $\ln(r)$ and has a singularity at $r = \infty$ (in other words propagator has a pole at zero momentum).

4.2 Ward-Takahashi Identities

We saw in last section, that our theory has gauge invariance built in at classical level *i.e.*, under gauge transformation, classical action and hence equation of motion remain invariant. In this section, we show that manifestation of gauge invariance at quantum level leads to Ward-Takahashi identities. These identities play pivotal role in proving renormalisability of a given gauge theory. Roughly, gauge invariance allows one to redefine the field variables without altering the physical content of the theory. So the value of vacuum persistence amplitude or generating functional Z

should be independent of any redefinition, hence we can write:

$$\delta_\Lambda Z = 0 = \int \mathcal{D}[B,\bar{\psi},\psi]\,\delta_\Lambda S\, e^{iS},$$
$$\text{when } \delta B_\mu = \partial_\mu \Lambda, \delta\psi = ig\Lambda\psi, \delta\bar{\psi} = -ig\Lambda\bar{\psi}. \tag{4.12}$$

These implies:

$$\int \mathcal{D}[B,\bar{\psi},\psi]\,\left[-\mu^2(\partial\cdot B) - \frac{1}{\xi}\partial^2(\partial\cdot B) + ig(\bar{\eta}\psi - \bar{\psi}\eta) - (\partial\cdot j)\right]\, e^{iS} = 0.$$

This can rewritten as:

$$\left(-\mu^2\partial_x^\mu \frac{\delta}{\delta j^\mu(x)} - \frac{1}{\xi}\partial_x^2\partial_x^\mu \frac{\delta}{\delta j^\mu(x)}\right) Z - (\partial\cdot j)Z = \left(-ig\bar{\eta}\frac{\delta}{\delta\bar{\eta}} + ig\eta\frac{\delta}{\delta\eta}\right) Z, \tag{4.13}$$

which is the master equation for identifying all identities amongst the various n-point functions of the theory.

Taking a functional derivative of above equation with respect to $j^\nu(y)$, and evaluating the expression with all sources going to zero, we get:

$$-\mu^2\partial_x^\mu D_{\mu\nu}(x-y) = \frac{1}{\xi}\partial_x^2\partial_x^\mu D_{\mu\nu}(x-y),$$

which in Landau gauge reads:

$$\partial_x^\mu D_{\mu\nu}(x-y) = 0.$$

This is one of the Ward-Takahashi identities of this gauge theory, and implies that, the longitudinal part of the gauge field propagator receives no corrections via fermion loops. In other words, if one starts from a transverse propagator at tree(classical) level, it would remain the same after quantum corrections too. This has important implications, in particular, we shall see that fermion loops can contribute a singular longitudinal part to gauge field propagator, which can be removed by regularisation on demanding validity of above identity.

Further, equation (4.13) can be rewritten in terms of what is called generating functional for connected diagrams W, where $Z = e^{iW}$ [101, 34], as:

$$-i\mu^2\partial_x^\mu \frac{\delta W}{\delta j^\mu(x)} - \frac{i}{\xi}\partial_x^2\partial_x^\mu \frac{\delta W}{\delta j^\mu(x)} = \partial\cdot j + g\bar{\eta}\frac{\delta W}{\delta\bar{\eta}(x)} - g\eta\frac{\delta W}{\delta\eta(x)}.$$

In terms of what is called Effective Action $\Gamma[B,\bar{\psi},\psi]$, which is defined as Legendre transform [101, 34] $\Gamma = W - \int d^3x \left(j\cdot B + \bar{\eta}\psi + \bar{\psi}\eta\right)$, above equation can be written as:

$$-\mu^2\partial_\mu^x B^\mu(x) - \frac{\partial_x^2}{\xi}\partial_\mu^x B^\mu(x) - ig\psi\frac{\delta\Gamma}{\delta\psi} + ig\bar{\psi}\frac{\delta\Gamma}{\delta\bar{\psi}} + \partial_\mu^x\frac{\delta\Gamma}{\delta B^\mu(x)} = 0.$$

Effective Action can be though of as quantum corrected action, where loop effects have been incorporated by appropriate terms in action itself. So the Greens function(propagator) that one finds using Effective Action is supposed to be exact Greens function of corresponding field, including all quantum corrections. By taking two functional derivatives of above equation, firstly with respect to $\bar{\psi}(y)$ and then followed by $\psi(x)$, and setting all the sources and fields to zero, one finds:

$$ig\delta^3(x-z)\frac{\delta^2\Gamma}{\delta\bar{\psi}(y)\delta\psi(x)} - ig\delta^3(x-y)\frac{\delta^2\Gamma}{\delta\bar{\psi}(x)\delta\psi(z)} + \partial^x_\mu\frac{\delta^3\Gamma}{\delta\psi(z)\bar{\psi}(y)\delta B_\mu(x)} = 0. \tag{4.14}$$

This is another Ward-Takahashi identity connecting the fermion two point function with the vertex function. This can rewritten in a slightly convenient way, by observing:

$$\frac{\delta\Gamma}{\delta\bar{\psi}(z)\delta\psi(y)} = S_F^{-1}(z-y),\ \Gamma^{(3),\mu}(z,y,x) = \frac{\delta^3\Gamma}{\delta\psi(z)\delta\bar{\psi}(y)\delta B_\mu(x)}, \tag{4.15}$$

as:

$$ig\delta^3(x-z)S_F^{-1}(y-x) - ig\delta^3(x-y)S_F^{-1}(x-z) + \partial^{x,\mu}\Gamma^{(3)}_\mu(z,y,x) = 0.$$

Here, $S_F(x-y)$ represents exact fermion propagator of this interacting theory, and $\Gamma^{(3)}_\mu(z,y,x)$ represents full vertex of the theory. In momentum space, it reads:

$$\Gamma^{(3)}_\mu(p,q) = g\frac{\partial}{\partial q^\mu}S_F^{-1}(p). \tag{4.16}$$

Above is a very powerful relation, and states that the knowledge of complete fermion propagator is sufficient to determine the full nonperturbative vertex of the theory. Notice, that while deriving this relation, no reference about value of coupling constant has been made, which makes this statement nonperturbative and remains valid to all orders. Interestingly, it also means that if one considers any quantum correction to electron propagator at certain order in perturbation, then to same order one needs to consider vertex correction.

4.3 Schwinger-Dyson equations

Full nonperturbative quantum dynamics of a given theory is described by a set of self consistent functional differential equations involving various n-point functions, and are called Schwinger-Dyson equations (SDE). These have been studied extensively in context of problems dealing with bound states [34], chiral symmetry breaking in QED_3 [73] and in many other problems. In most cases it is not possible to solve these equations exactly, and various approximation are usually sought to solve these.

The SDE involving fermion two point function can be easily arrived by noting that [34]:

$$\int \mathcal{D}\bar{\psi}_+ \frac{\delta Z}{\delta \bar{\psi}_+} = 0,$$

which actually is a consequence of translational invariance of measure in above functional integral. This implies in turn that:

$$\int \mathcal{D}[\bar{\psi}_+, \psi_+, B_\mu] \, \frac{\delta S}{\delta \bar{\psi}_+} \, e^{iS} = 0$$

$$\text{or} \int \mathcal{D}[\bar{\psi}_+, \psi_+, B_\mu] \, \left[(i\not{\partial}_+ - m)\psi_+ + \eta\right] \, e^{iS} = 0.$$

In terms of Z above can be rewritten as:

$$(-i)\left[i\gamma_+^\mu \partial^\mu - m - ig\gamma^\mu{}_+ \frac{\delta}{\delta J^\mu}\right] \frac{\delta Z}{\delta \bar{\eta}} + \eta Z = 0.$$

Taking a derivative with respect to $\eta(y)$, the above equation then reads:

$$\delta^3(x-y) + \left[i\gamma_+^\mu \partial_\mu^x - m - ig\gamma_+^\mu \frac{\delta}{\delta J^\mu(x)}\right] S_F(y-x) = 0. \tag{4.17}$$

Functional derivative of $S_F(x-z)$ with respect to $J^\mu(x)$ can be easily calculated to give:

$$\frac{\delta S_F(x-z)}{\delta J^\mu(x)} = -\int d^3y \, d^3u \, d^3v \, D_{\mu\nu}(x-y) S_F(x-u) \Gamma^\nu_{(3)}(v,u,y) \, S_F(v-z),$$

where $\Gamma^\nu_{(3)}(v,u,y)$ is the vertex function given by $\frac{\delta^3 \Gamma}{\delta\psi(v)\delta\bar{\psi}(u)\delta B_\nu(y)}$. Using above expression, equation (4.17) can be written as:

$$S_F^{-1}(x-w) = (i\not{\partial}_+^x - m)\delta^3(x-w) + ig\gamma_+^\mu \int d^3z \int d^3u \, D_{\mu\nu}(x-z) \, S_F(x-u) \, \Gamma^{\nu,(3)}(w,u,z), \tag{4.18}$$

which the Fermion SDE.

The SDE corresponding to gauge field two point function can be similarly derived, and is given as:

$$\begin{aligned}[D^{\mu\nu}(x-y)]^{-1} = &\left((1-\frac{1}{\xi})\partial^\mu\partial^\nu - \eta^{\mu\nu}(\partial^2+\mu^2)\right)\delta^3(x-y) \\ &+ ig\mathrm{Tr}\int d^3u \int d^3v \, \gamma^\mu S_F(u-x) \, \Gamma^\nu_{(3)}(y;u,v) \, S_F(v-x)\end{aligned} \tag{4.19}$$

where, $\Gamma^\nu_{(3)}(y;u,v) = \frac{\delta^3\Gamma}{\delta B_\nu(y)\delta\psi(u)\delta\bar{\psi}(v)}$ and Tr stands for trace of Dirac matrices. This equation can be approximately solved, using perturbative approximation, by considering tree level vertex and

free propagators in the last term of the right hand side of above equation. In Fourier space, this can be conveniently written as:

$$(D^{\mu\nu}(p))^{-1} = (D_0^{\mu\nu}(p))^{-1} + \Pi^{\mu\nu}(p),$$
$$\text{where } \Pi^{\mu\nu}(p) = ig^2 \text{Tr} \int \frac{d^3k}{(2\pi)^3} \gamma^\mu \frac{1}{\not{k} - m} \gamma^\nu \frac{1}{\not{k} + \not{p} - m}.$$

A straightforward calculation, as previously, encountered yields:

$$\Pi^{\mu\nu}(p) = -\frac{g^2}{\pi^2} \int_0^1 dx \left[\frac{\eta^{\mu\nu}}{3} \int_0^\infty dk + \frac{x(x-1)\pi}{2\sqrt{Q^2}} (\eta^{\mu\nu} p^2 - p^\mu p^\nu) - \frac{im\pi}{4\sqrt{Q^2}} \epsilon^{\mu\nu\lambda} p_\lambda \right], \tag{4.20}$$

where $Q^2 = x(x-1)p^2 + m^2$. Using gauge invariant Pauli-Villars regularisation scheme, where one couples additional pair of heavy fermions to the gauge particle, and subtracts out the contribution arising from the these fermions to gauge propagator [34]. This amounts to calculate regularised $\Pi_{\mu\nu}$ as:

$$\Pi^R_{\mu\nu}(p) = \Pi_{\mu\nu}(p; g, m) + c_1 \Pi_{\mu\nu}(p; g, \Lambda_1) + c_2 \Pi_{\mu\nu}(p; g, \Lambda_2),$$

where $\Lambda_{1,2}$ are masses of heavy fermions, and $c_{1,2}$ are constants. One can clearly see, that choosing $c_1 + c_2 + 1 = 0$ allows one to get rid of divergence. In above calculation, we have only considered coupling of K_+ electrons to phonons, but we know for sure that K_- electrons also couple to gauge field and phonon self energy would invariably receive contribution from K_- fermion loops. Importantly, as noted before the Gamma matrices for both differ by a minus sign and so the trace of odd Gamma matrices too. This results in cancellation of parity odd term in self energy expression (4.20), yielding:

$$\begin{aligned} \Pi^{\mu\nu}(p) &= \Pi(p)(\eta^{\mu\nu} p^2 - p^\mu p^\nu) \\ &= \left[-\frac{g^2}{\pi^2} \int_0^1 dx \frac{x(x-1)\pi}{\sqrt{Q^2}} \right] (\eta^{\mu\nu} p^2 - p^\mu p^\nu). \end{aligned}$$

When $p^2 < 4m^2$, the integral can be evaluated to give:

$$\Pi(p) = \frac{-g^2}{2\pi p^2} \left[|m| + \frac{4m^2 + p^2}{4|p|} \ln \left(\frac{2|m| - |p|}{2|m| + |p|} \right) \right].$$

On the other hand, when $p^2 > 4m^2$, $Q^2 < 0$ and hence a branch cut opens up starting from two particle threshold $p^2 = 4m^2$. This leads to a finite imaginary part of $\Pi(p)$:

$$\begin{aligned} \Pi(p) &= \Pi_R(p) + i\Pi_I(p) \\ &= \frac{-g^2}{2\pi p^2} \left[|m| + \frac{4m^2 + p^2}{4|p|} \ln \left(\frac{-2|m| + |p|}{2|m| + |p|} \right) + i \frac{\pi(4m^2 + p^2)}{4|p|} \right]. \end{aligned}$$

The one loop renormalised gauge field propagator can be written in Landau gauge as:

$$D_{\mu\nu}(p) = -\frac{1}{(1-\Pi^{reg}(p))p^2 - \mu^2}(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}).$$

In case, when the external gauge boson momentum is greater than two particle threshold, it can lead to fermion-antifermion pair creation, with the decay width (which is inverse of lifetime)

$$\Gamma = -\frac{\mu\Pi_I^{reg}(\mu)}{(1-\Pi_R^{reg}(\mu))^2 + (\Pi_I^{reg}(\mu))^2}. \tag{4.21}$$

As is obvious, gauge field mass gets renormalised due to interactions, and renormalised physical mass is given by

$$\mu_{ph}^2 = \frac{\mu^2(1-\Pi_R^{reg}(\mu))}{(1-\Pi_R^{reg}(\mu))^2 + (\Pi_I^{reg}(\mu))^2}. \tag{4.22}$$

4.4 Electromagnetic response

As was done in the models discussed earlier, our goal is to study the electromagnetic response of graphene electrons interacting with gauge fields. A convenient way to serve the purpose is by looking at low energy effective action for external electromagnetic field, which is obtained by integrating over all the dynamical fields. As was done in the previous models, we consider the situation when the fermions mass is large, so that they can be readily integrated out to yield an effective action comprising of dynamical gauge field B_μ and external electromagnetic field using the derivative expansion scheme used earlier. Since, both electromagnetic field A_μ and B_μ couple to Dirac fermions minimally, this can be easily evaluated at one loop order to read:

$$\mathscr{L}_{eff}[B,A] = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{\mu^2}{2}B_\mu B^\mu - \frac{mg}{\pi|m|}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu B_\lambda.$$

Except the gauge field mass term rest of the terms are essentially the same as that seen in previous models. Infact it is this precise term whose effect on electromagnetic response we are interested in. Dynamical gauge field B_μ in above action can be integrated out to give an effective action for A_μ:

$$\mathscr{L}_{eff}[A] = \frac{-g^2}{4\pi^2}F_{\mu\nu}\frac{1}{\partial^2+\mu^2}F^{\mu\nu}.$$

Note the presence of massive pole in above action, which immediately indicated that this action does not describe the response of a superconductor. Inorder to understand the response it is convenient to find out the induced current due the external electromagnetic field, to which the

external electromagnetic field itself couples. Induced current can be easily found from the above action to be:

$$J^{\nu} = -\frac{g^2}{\pi}\frac{1}{\partial^2 + \mu^2}\partial_{\mu}F^{\mu\nu}.$$

This couples to the electromagnetic field which is governed by Maxwell's equation of motion:

$$\partial_a F^{a\mu} = e^2\delta(z)J^{\mu}.$$

Electromagnetic response of a given material is supposed to be captured by its electric and magnetic susceptibilities [103]. Since essentially we are looking at low energy linear response of graphene in presence of this massive Abelian gauge field, above equations easily allows us to determine the susceptibilities which are given by:

$$\epsilon(w) = \mu(w)^{-1} = 1 - \frac{\delta(z)g^2e^2}{\pi(w^2 - \mu^2 + i\mu\gamma)}.$$

Here electric/magnetic susceptibility as a function of frequency w is given by $\epsilon(w)/\mu(w)$. In above expression we have also added a phenomenological decay width γ for the gauge particle, since in most real situations it may not be long lived. This makes the susceptibility complex and paves the way for absorption/emission of photons. Infact it can be seen that, for suitable choice of constants, around $w = \mu$ real part of both electric and magnetic susceptibility changes sign, whereas the imaginary part peaks. This may remind one of metamaterial or material with negative refractive index. However, since $\epsilon(w) = \mu(w)^{-1}$, this possibility negated and the refractive index $n(w) = \sqrt{\epsilon(w)\mu(w)}$ remains essentially positive. However, the coupled behaviour of ϵ and μ is very unique and strange in some sense. The former tells us about the charge separation in the system, whereas the latter about currents and spin fluctuations. Their sychronized behaviour can be traced back to the fact that due to fermion loop correction, B_{μ} field actually becomes electrically charged. Further, it is massive and hence as per our earlier discussion it carries spin. However, it does not violate parity or time reversal invariance. It has been shown by Deser that, massive vector field actually has two degrees of freedom, one with spin $+1$ and the other with -1, as a result of which this is possible [131]. It can be seen that Deser's argument will also hold for massive gauge field B_{μ}. Hence not only ϵ but also μ becomes singular at $w = \mu$, where the gauge particles are created, which carry an effective charge and spin. Observation of such a correlated behaviour of these susceptibilities will confirm, without any doubts, existence of fermion spin, since they are intermediate particles in the process.

Chapter 5

Femtosecond pulse propagation in optical fibres

More than hundred years back, Maxwell proposed his theory of electrodynamics, and unified otherwise different subjects of electricty and magnetism. The equation governing propagation of electromagnetic fields in vacuum, called Maxwell's equations are given by:

$$\begin{aligned}\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}&=0,\\ \vec{\nabla}\times\vec{B}&=\frac{\partial\vec{E}}{\partial t},\\ \vec{\nabla}\cdot\vec{E}&=0\\ \vec{\nabla}\cdot\vec{B}&=0.\end{aligned}$$

Maxwell himself thought that these fields are undulations of some medium called *luminiferous ether* which is all pervading, much in analogy to sound waves [132]. The effect of material objects which are influenced by these fields or can influence these fields, is taken into account by modifying Maxwell's equations which are given (in SI units) by [103]:

$$\begin{aligned}\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}&=0,\\ \vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t},&\\ \vec{\nabla}\cdot\vec{D}&=\rho_f,\\ \vec{\nabla}\cdot\vec{B}&=0.\end{aligned}$$

Here, $\vec{E}$ and $\vec{H}$ are electric and magnetic field vectors respectively, whereas $\vec{D}$ and $\vec{B}$ are electric and magnetic flux densities. The current density $\vec{J}$, and charge density ρ represent coupling of free charged particles with electromagnetic fields, and are zero in a metal and in a dielectric media like glass. Flux densities owe their origin to response of medium under influence of electromagnetic fields, and are defined as:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}, \;\; \vec{B} = \mu_0 \vec{H} + \vec{M},$$

where ϵ_0 & μ_0 represent permittivity and permeability of vacuum respectively, and $\vec{P}$ and $\vec{M}$ represent induced electric and magnetic polarisations. Maxwell thought that by explicit introduction of permittivity and permeability of vacuum, one has explicitly proven existence of ether, since it is suppose to represent its physical properties. However after an intense intellectual struggle, we now understand that luminiferous ether is non existent. Nevertheless both permittivity and permeability of vacuum are reality, and one wonders what 'media' is responsible for its origin. After the advent of quantum field theory it became clear that, vacuum of a given theory behaves like a medium, whose polarisability can be calculated and measured experimentally. This showed that vacuum in reality is not truly empty but is a vast store of virtual particles or quantum fluctuations. In presence of a material medium, these obviously change, which changes the polarisability and leads to change in the usual behaviour of electromagnetic fields. As we did in previous chapters, with sufficient knowledge of microscopic interaction, one can in principle derive polarisability by calculating vacuum polarisation tensor. In most cases, it is not possible to calculate the vacuum polarisation tensor exactly and one often estimates the same in low energy limit. This is done by separating out high energy or 'fast' modes of the theory and integrating them out to get a low energy effective theory for 'slow' modes. This technique does work for many systems of interest, but fails if the medium contains too many excitations or simply if the interactions are strongly coupled and do not have enough symmetry.

In such case, it is useful to use a phenomenological approach and based on an intutive symmetry-based argument propose a broad ansatz for polarisability. Plugging this in the equation of motion will yield electromagnetic response of the given system which can be compared with the experiments to, first, determine unknown parameters if any in the theory, and then predict new phenomena based on the same. This approach certainly lacks the knowledge of microscopic working, but in the spirit of Landau-Ginzburg theory, can model and encompass a large class of systems under its umbrella. In this chapter, we shall be working with this philosophy and will confine ourselves to a

nonmagnetic dielectric media, where $\vec{M} = 0$. Using above relations and Maxwell's equations, one can easily obtain an equation only in terms of $\vec{E}$ and $\vec{P}$:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}, \tag{5.1}$$

where c is speed of light in vacuum and is given by $c^2 = \frac{1}{\epsilon_0 \mu_0}$. Now inorder to study propagation of electromagnetic fields inside a given dielectric media, one needs to know the relation between $\vec{P}$ and $\vec{E}$. We phenomenologically write polarisation as made up of linear and nonlinear parts:

$$\vec{P} = \vec{P}_L + \vec{P}_{NL},$$

where,

$$\vec{P}_L(\vec{r}, t) = \epsilon_0 \int_{-\infty}^{\infty} dt' \, \chi_1(t - t')\vec{E}(\vec{r}, t'), \tag{5.2}$$

and

$$\vec{P}_{NL}(\vec{r}, t) = \epsilon_0 \int\int\int_{-\infty}^{\infty} dt_1 dt_2 dt_3 \, \chi_3(t - t_1, t - t_2, t - t_3) \vdots \vec{E}(\vec{r}, t_1)\vec{E}(\vec{r}, t_2)\vec{E}(\vec{r}, t_3), \tag{5.3}$$

where $\chi_{1,3}$ represent first and third order susceptibilities, and higher order susceptibility contribution is ignored. The three dots in above expression for $\vec{P}_{NL}$ represent tensor product. While writing above expressions, we have assumed that material response is local and only electric dipole processes contribute to it. Also we have assumed that nonlinear response is solely due to third order response. Equipped with above expressions, we are now in a position to study wave propagation in a nonlinear dielectric media.

In this chapter, we shall discuss the problem of pulse propagation in an optical fibre, which is nothing but a cylindrical structure made up of two dielectric media. This system is axially symmetric, and below what is called the core radius, one kind of medium exists with refractive index n_1, and above it exist another kind of medium with refractive index n_2. It is trivial to see that if $n_1 < n_2$, then due to total internal reflection, it is possible that any incoming radiation from one end of fibre with suitable frequency can be 'trapped' inside the core region can be guided another end [103]. As is evident, optical transmission is such system would be extremely efficient since optical losses can be minimised. Optical fibre communication is now a reality, and almost all of intercontinental internet traffic or telephonic traffic goes via them. Ease of handling optical fibres, their ability to be flexible and robustness against external conditions makes them very useful for any reliable communication system. A given digital electric al signal is converted into an optical

one using a laser, which typically operates in micrometer range. Logical OFF is represented by small almost zero background intensity, whereas logical ON is represented by a light pulse of certain height and width. The sensitivity of receiver at the other end, decides what pulse height needs to be fed in so that one has a good signal-to-noise ratio. On other hand, how much information one wants to send in a unit time is what fixes the pulse width. The smaller the width, the more pulses can be sent in a given time interval, conveying more information. But shorter pulses imply that they involve a broad frequency range and response of optical fibre may not be same over all of these. Further, inorder that this short pulses are clearly detectable, they must be taller to have sufficient energy to register a 'click' in the detector. Now taller pulse have a larger peak power and hence peak intensity. Now it is in this domain that above mentioned nonlinear polarisability plays an important role [10]. In most cases, it is observed that nonlinear contribution to polarisation is much smaller compared to the linear contribution. So in what follows we shall consider the nonlinear effects only perturbatively over linear contribution. Next section deals with this simpler problem of pulse propagation in an optical fibre whose response is linear. This is will be followed by a nonlinear version of the same problem in two regimes, first in nanosecond and latter in femtosecond.

5.1 Optical fibre: Linear regime

In the linear regime, frequency dependent dielectric constant $\epsilon(\omega)$ is defined as $\epsilon(\omega) = 1 + \tilde{\chi}_1(\omega)$, where $\tilde{\chi}_1(\omega)$ is Fourier transform of $\chi_1(t)$. So frequency domain, this relation along with (5.2) allows one to write (5.1) as:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{\mathscr{E}}(\vec{r}, \omega) - \frac{\epsilon(\omega)\omega^2}{c^2} \vec{\mathscr{E}}(\vec{r}, \omega) = 0, \tag{5.4}$$

where $\vec{\mathscr{E}}(\vec{r}, \omega)$ is Fourier transform of $\vec{E}(\vec{r}, t)$ defined as:

$$\vec{\mathscr{E}}(\vec{r}, \omega) = \int_{-\infty}^{\infty} dt\, \vec{E}(\vec{r}, t)\, e^{i\omega t}.$$

Dielectric constant $\epsilon(\omega)$ is in general complex, and it can be shown that the imaginary part contributes to loss. In this discussion, however we will ignore loss term to keep the treatment simple, however it can always be taken into account phenomenologically at later stage. This allows one to write refractive index $n^2(\omega) = \epsilon(\omega)$, and further we consider it to independent of $\vec{r}$ so that $\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot \vec{E} = 0$. With these considerations, above equation reduces to a simple eigenvalue

problem:

$$\nabla^2\vec{\mathscr{E}} + \frac{n^2(\omega)\omega^2}{c^2}\vec{\mathscr{E}} = 0.$$

Using the cylindrical symmetry of the fibre it is convenient to work in cylindrical coordinates ρ, ϕ and z, in which case above equation reads:

$$\frac{\partial^2\vec{\mathscr{E}}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\vec{\mathscr{E}}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\vec{\mathscr{E}}}{\partial\phi^2} + \frac{\partial^2\vec{\mathscr{E}}}{\partial z^2} + n^2k_0^2\vec{\mathscr{E}} = 0,$$

where, $k_0 = \frac{\omega}{c}$. It is easy to see that a similar equation will also be obeyed by $\vec{H}$. Since, both $\vec{E}$ and $\vec{H}$ are connected by Maxwell's equations, only two of six degrees of freedom are independent. It is convenient to choose these two independent variables to be E_z and H_z, and whose Fourier transforms $\mathscr{E}_z$ and $\mathscr{H}_z$ obey above equation. Above equation in $\mathscr{E}_z$ can be easily solved using the method of separation of variables, to give solution of the form:

$$\mathscr{E}_z(\vec{r},\omega) = A(\omega)F(\rho)e^{\pm im\phi}e^{i\beta z},$$

where β is the propagation constant, A is normalisation constant, m is an integer, and $F(\rho)$ is a function that solves

$$\frac{d^2F}{d\rho^2} + \frac{1}{\rho}\frac{dF}{d\rho} + \left(n^2k_0^2 - \beta^2 - \frac{m^2}{\rho^2}\right)F = 0.$$

Refractive index in above equation equals to n_1 for all $\rho \leq a$ and equals to n_2 for all $\rho > a$. As is evident, this differential equation is the one that is solved by Bessel functions. Hence it is easy to show that, non-singular solutions in core region $\rho \leq a$ are given by:

$$F(\rho) = J_m(\kappa\rho),$$

where $\kappa^2 = (n_1^2k_0^2 - \beta^2)$. We seek solutions in cladding region $\rho > a$ such that they decay for large values of ρ, that is to say that they are guided modes. Under such consideration, above equation yields solution for cladding region as

$$F(\rho) = K_m(\gamma\rho),$$

$\gamma^2 = (\beta^2 - n_2^2k_0^2)$. Same procedure can be followed for H_z component. Matching conditions on field inside and outside the core leads to determination of β. In general there are several solutions for β, each representing a different fibre mode. In the present discussion we shall confine ourselves to case where is only one stable propagating mode denoted by HE_{11} and called the fundamental mode. Electric field distribution corresponding to this mode has all three components non-zero, amongst

which either E_x or E_y dominate. Assuming that the fibre is such that it maintains polarisation of light intact, electric field for the fundamental mode can be approximately written as:

$$\vec{\mathscr{E}}(\vec{r},\omega) = \hat{x}\, A(\omega)F(x,y)e^{i\beta(\omega)z},$$

where A is normalisation constant. Transverse distribution in core region is given by:

$$F(x,y) = J_0(\kappa\rho),$$

whereas in the cladding it goes as

$$F(\rho) = \sqrt{\frac{a}{\rho}}J_0(\kappa a)e^{-\gamma(\rho-a)}.$$

Using matching conditions at $\rho = a$, one obtains an equation in β. In most case, it is difficult to find exact solutions for β, in which case one numerically solves the equation.

5.2 Nonlinear regime

In most situations, one is concerned with pulse propagation of in optical fibres. if the pulse width is of the order of nanoseconds or less, then nonlinear contribution to polarisation becomes important. In that case, the pulse propagation is governed by following equation:

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial^2\vec{P}_L}{\partial t^2} + \mu_0\frac{\partial^2\vec{P}_{NL}}{\partial t^2}.$$

Nonlinear change in refractive index is usually small of the order of $< 10^{-6}$, and hence P_{NL} can be treated perturbatively. As in earlier case, here also consider the situation where polarisation of pulse does not change as it propagates through the fibre, so that essentially a scalar treatment holds. Optical field is assumed to be quasi-monochromatic *i.e.,* pulse spectrum is centered around ω_0 and has a width $\delta\omega$ such that $\frac{\delta\omega}{\omega_0} << 1$. In reality $\omega_0 \sim 10^{15}\, Hz$ so this assumption holds for pulses as short as 0.1 ps. This allows us to separate the fast moving component of field and relatively slowly moving component of field, which is known as slowly varying envelope approximation in literature. So,

$$\vec{E}(\vec{r},t) = \frac{1}{2}\hat{x}\left(E(\vec{r},t)e^{-i\omega_0 t} + c.c.\right),$$

where $\hat{x}$ is the polarisation vector, $E(\vec{r},t)$ is slowly varying function in time relative to $e^{i\omega_0 t}$. Similar decomposition also holds for both P_L and P_{NL}. This allows one to write:

$$\begin{aligned} P_L(\vec{r},t) &= \epsilon_0 \int_{-\infty}^{\infty} dt'\, \chi_1(t-t')\, E(\vec{r},t') e^{i\omega_0(t-t')} \\ &= \frac{\epsilon_0}{2\pi} \int_{-\infty}^{\infty} d\omega\, \tilde{\chi}_1(\omega)\, \mathscr{E}(\vec{r},\omega) e^{-i(\omega-\omega_0)t}. \end{aligned}$$

Treatment of P_{NL} considerably simplifies if one assumes that nonlinear responses are all instantaneous. This essentially implies that one is ignoring Raman processes which occur over time scale ~ 60 fs, and is only valid for pulses of width more than few ps. Hence, one may write,

$$P_{NL}(\vec{r},t) = \epsilon_0 \epsilon_{NL} E_{\vec{r},t},$$

where nonlinear contribution to ϵ is given by

$$\epsilon_{NL} = \frac{3}{4}\chi_3 |E(\vec{r},t)|^2.$$

To obtain a wave equation for slowly varying field E, it is convenient to work in Fourier domain as in earlier case. However, the presence of nonlinear term in ϵ does not allow a local equation in Fourier domain. In this approach, since anyway we have assumed E to be a slowly varying function, we shall consider ϵ_{NL} as constant while going to Fourier space, to get a Helmholtz like equation:

$$\nabla^2 \mathscr{E} + k_0^2 \epsilon(\omega) \mathscr{E} = 0,$$

where $\epsilon(\omega) = 1 + \tilde{\chi}_1(\omega) + \epsilon_{NL}(\omega)$. This equation can be solved using method of separation of variables, and we assume solution of the form:

$$\mathscr{E}(\vec{r},\omega) = F(x,y)\tilde{A}(z,\omega-\omega_0)e^{i\beta_0 z},$$

where $A(z,\omega)$ is a slowly varying function and $\beta_0(\omega)$ is wave number which is to be determined. Substituting this solution in above equation, gives rises to two equations, one in F:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \left(\epsilon(\omega)k_0^2 - \tilde{\beta}^2\right) F = 0,$$

and the other in $\tilde{A}$:

$$2i\beta_0 \frac{\partial \tilde{A}}{\partial z} + \left(\tilde{\beta}^2 - {\beta_0}^2\right) \tilde{A} = 0.$$

Wave number $\tilde{\beta}$ is determined by solving the above equation in F. Dielectric constant can be approximated as $\epsilon(\omega) = (n+\delta n)^2 \approx n^2+2n\delta n$, where δn is small perturbation given by $\delta n = n_2|E|^2$. One first solves the equation in F assuming δn to be zero, and obtains modal distribution function $F(x,y)$ and $\beta(\omega)$. The effect of δn is now included perturbatively, and it can be shown that to the first order, $F(x,y)$ does not change, only the eigenvalue β changes to $\tilde{\beta}(\omega) = \beta(\omega) + \delta\beta$, where

$$\delta b = \frac{k_0 \int\int dxdy\, \delta n |F|^2}{\int\int dxdy\, |F|^2}.$$

This allows us to write equation in $\tilde{A}$ as:

$$\frac{\partial \tilde{A}}{\partial z} = i\left(\beta(\omega) + \delta\beta - \beta_0\right)\tilde{A},$$

which clearly tells the each spectral component acquires a phase shift which is both intensity and frequency dependent. By performing inverse Fourier transform of above equation, one obtains an equation governing dynamics of slowly varying part of electric field along the fibre axis. However, as noted earlier the precise form of β is seldom known, and hence it is useful to Taylor expand it around the carrier frequency ω_0:

$$\beta(\omega) = \beta_0 + (\omega-\omega_0)\beta_1 + \frac{1}{2}(\omega-\omega_0)^2\beta_2 + \cdots$$

and terms beyond quadratic order since we have assumed that the pulse is quasi-monochromatic. With this expansion, it is easy to invert above equation to get an equation in time domain as:

$$\frac{\partial A}{\partial z} + \beta_1\frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = i\gamma|A|^2A, \tag{5.5}$$

where nonlinearity parameter γ is defined as $\gamma = \frac{n_2\omega_0}{cA_{eff}}$, where effective core area is defined as $A_{eff} = \frac{\left(\int\int dxdy|F|^2\right)^2}{\int\int dxdy|F|^4}$. In a comoving frame $T = t - \beta_1 z$, above equation reduces to:

$$i\frac{\partial A}{\partial z} - \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \gamma|A|^2A = 0, \tag{5.6}$$

which is the famous nonlinear Schrodinger equation (NLSE). This system is one of the well studied models in literature. It was first predicted by Hasegawa and Tappert [11], that this equation admits robust localised solutions called solitons. These are truly nonperturbative solutions of above equation, and are stable against collisons *i.e.,* two solitons propagating in opposite directions collide, interact strongly before emerging out intact without exchanging energy and momentum with one another. These were later experimentally observed by Mollenauer *et al.* in optical fibres[133].

The origin of these solitons lies in existence of an infinite dimensional symmetry algebra, which is responsible for its Liouville integrability [134]. Existence of such huge symmetry group for above dynamical system is not surprising, since it is known that many systems in 1+1 dimensional space-time are integrable, Korteweig-de Vries equation and sine-Gordon model being famous examples. A unifying feature amongst these integrable models is the fact that their symmetry algebra can be connected to conformal algebra in 1+1 dimensions, which itself is infinite dimensional. This gives rise to various mathematical structures and there are various formalisms for studying these. Formalisms like inverse scattering transform, Hirota method, Sato theory, zero-curvature formalism, Backlund-Darboux transforms, Lie-symmetry based formalisms, phase space methods like bi-Hamiltonian structures are the most popular ones amongst others [135, 136, 137].

5.3 Higher order nonlinear effects

With the advent of high intensity laser beams, it has become possible to generate optical pulses with width of the order of 10 femtoseconds. In such situations, many of the approximations considered in arriving at above equation become questionable. The most crucial one is that now no longer polarisation effects can be considered instantaneous, since Raman effect becomes important at these time scales. As done earlier, one can phenomenonlogically model these retarded contributions to polarisation, in case when the radiation is of non-resonant and incoherent, by assuming third order susceptibility as:

$$\chi_3(t-t_1, t-t_2, t-t_3) = \chi_3 R(t-t_1)\delta(t-t_2)\delta(t-t_3),$$

where $\int dt R(t) = 1$, and $R(t-t')$ is zero for all $t' > t$ to ensure causality. Nonlinear polarisation in this case is given by:

$$\vec{P}_{NL}(\vec{r},t) = \epsilon_0 \chi_3 \vec{E}(\vec{r},t) \int_{-\infty}^{t} dt'\, R(t-t')|\vec{E}(\vec{r},t')|^2.$$

With this redefinition of polarisation, one can still use the technique of going to frequency domain, as in previous case, and one obtain a femtosecond generalisation of above NLSE:

$$\begin{aligned} iA_z + a_1 A_{tt} + a_2|A|^2 A \\ +i\left[a_3 A_{ttt} + a_4(|A|^2 A)_t + a_5 A(|A|^2)_t\right] = 0, \end{aligned} \tag{5.7}$$

which contains a third order dispersion with coefficient a_3, a self steepening term with coefficient $a_4 = a_2/\omega_0$ and self frequency shift effect coming from Raman scattering with coefficient a_5, which equal to the first moment of $R(t)$. This was first obtained by Kodama [138] and Kodama & Hasegawa [139], using rigourous multiscale pertubation theory which justifies our semi-rigourous arguments. In what follows we shall assume that above equation is appropriately scaled using relevant dimensional parameters and is now in dimensionless form. This model, unlike NLSE, is not integrable in general. A few integrable cases have been identified: (i) Sasa-Satsuma case (a_3:a_4:($a_4 + a_5$) = 1:6:3) [140], (ii) Hirota case (a_3:a_4:($a_4 + a_5$) = 1:6:0) [141] and (iii) derivative NLSE of type I and type II [142]. Many restrictive special solutions of bright and dark type have been obtained [143, 144, 145].

As mentioned earlier, that the effect of third order dispersion is significant for femtosecond pulses when group velocity dispersion (a_2) is close to zero. It is negligible for optical pulses whose width is of the order of 100 femtoseconds or more, having power of the order of 1 Watt and being operated in region where a_2 far away from zero. However, in this case self steepening, as well as self frequency shift terms are still dominant and should be retained. The effects of these higher order terms on pulse propagation have been extensively studied numerically [10, 146], and some special solutions to this system are also known [147]. With a non-integrable model at hand it in general a difficult task to obtain solutions to above equation of motion. We shall adopt here a very simple and effective ansatz based method. Essentially this method helps one to identify few special solutions which may help to built an insight about the dynamics of the system. A very similar technique has been used to solve equation of motion of SU(2) Yang-Mills Chern-Simons theory, which too are notoriously nonlinear and the system in general is not integrable (see Appendix).

Modulo a trivial kinematic phase, the complex envelope travelling wave solutions can be generally represented as:

$$A(x,t) = \rho(\xi)e^{i\chi(\xi)} \tag{5.8}$$

where $\xi = \alpha(t - ux)$ is the travelling coordinate, and ρ and χ are real functions of ξ. Here, α is scale parameter and $u = 1/v$ with v being the group velocity of the wave packet. The ansatz solution leads to the compatibility conditions:

$$-\alpha u\rho' + 2\alpha^2 a_1\chi'\rho' + \alpha^2 a_1\chi''\rho$$
$$+3\alpha a_4\rho^2\rho' + 2\alpha a_5\rho^2\rho' = 0, \tag{5.9}$$
$$\alpha u\chi'\rho + \alpha^2 a_1\rho'' - \alpha^2 a_1\chi'^2\rho + a_2\rho^3 - \alpha a_4\chi'\rho^3 = 0. \tag{5.10}$$

Equation (5.9) can be exactly integrated to yield:

$$\chi' = \frac{u}{2\alpha a_1} + \frac{c}{\alpha a_1\rho^2} - \frac{(3a_4 + 2a_5)}{4\alpha a_1}\rho^2, \tag{5.11}$$

where c is to be determined by initial conditions. It is to be noted that the phase has a nontrivial form and has two intensity dependent chirping terms, apart from kinematic first term. As is evident the second term is of kinematic origin and is common to Schrödinger equation as well. The last term is due to higher nonlinearities and leads to chirping that is exactly inverse to that of the former. This is a novel form of self phase modulation which is controlled by interaction. The amplitude equation (5.10) reduces to:

$$\theta_1\rho'' + \theta_2\rho + \theta_3\rho^3 + \theta_4\rho^5 = \frac{c^2}{\rho^3}$$

with $\theta_1 = \alpha^2 {a_1}^2$, $\theta_2 = \frac{(u^2 - ca_4 + 2ca_5)}{4}$, $\theta_3 = \frac{(2a_1a_2 - ua_4)}{2}$ and $\theta_4 = \frac{(4a_4 - 1)(3a_4 + 2a_5)}{16}$. We note that the nontrivial contribution from higher order nonlinear terms is through θ_4, which is zero for $a_4 = 1/4$ or $a_4 : a_5 = -2 : 3$ (assuming $a_4 \neq 0$ and $a_5 \neq 0$), and as we shall soon show, this results in interesting physical consequences. In the case when $a_4 : a_5 = -2 : 3$, both the intensity as well as phase will not be having any new features due to higher order nonlinearities and the solutions will exactly resemble NLSE solutions. However unlike NLSE both dark and bright solitons exists, in both normal and anomalous dispersion regime. When $c = 0$, existence of fundamental bright solitons with $\rho = A\text{sech}(\xi)$ is forbidden, and only dark solitons with $\rho = A\tanh(\xi)$ exist, with $\alpha^2 = \frac{u^2}{2{a_1}^2}$. Furthermore, these dark solitons in anomalous dispersion regime, i.e., $a_1 > 0$, respect the inequality $a_4 u \geq 2|a_1||a_2|$, which restrict them to travel only along one direction, given by sign of a_4. This is an example of chiral soliton which is absent in NLSE. When $c \neq 0$, both bright soliton and dark soliton exists, and satisfy $2a_1a_2 > ua_4$ and $2a_1a_2 < ua_4$ respectively, showing that both have mutally exclusive velocity space. As a consequence, in anomalous dispersion regime, dark solitons obey the inequality $2|a_1||a_2| < ua_4$ hence are chiral, whereas in normal dispersion regime bright solitons satisfy $-2|a_1||a_2| > ua_4$ and hence are also chiral. Notice that the directionality

of these solitons is due to the presence of higher order terms, the sign of a_4 decides the direction in which solitons are allowed to propagate. For $a_4 = 1/4$, the intensity profile will be the same as NLSE, whereas phase will still show nontrivial chirping. In this case also both dark and bright solitons can be chiral, and can exist in normal and anomalous disperison regimes, which is in sharp contrast to NLSE.

It is very intriguing to see that when $a_4 = \frac{-2a_5}{3}$ and $u = \frac{-3a_1a_2}{a_5}$, equation (5.12) combined with equation (5.11) reduces to free particle Schrödinger equation in ψ. So for this choice of parameters, in the presence of both Kerr and higher order nonlinearities, the effective evolution equation for ψ is linear, and one would expect to see phenomena like interference, which is forbidden otherwise in this system.

Equation (5.12) can be cast into a convenient form using $\rho = \sqrt{\sigma}$:

$$\frac{\theta_1}{2}\sigma'' + 2\theta_2\sigma + \frac{3\theta_3}{2}\sigma^2 + \frac{4\theta_4}{3}\sigma^3 = k, \tag{5.12}$$

where k is constant fixed by initial conditions. Solutions for this equation, with $\theta_4 \neq 0$, can be found by conformal Möbius transformation:

$$\sigma = \frac{A + Bf}{C + Df}, \tag{5.13}$$

which for some suitable A, B, C and D, connects σ to elliptic function f. These elliptic functions, as is known, are generalization of trigonometric and hyperbolic functions and appear in solutions of many nonlinear equations.

Considering the importance of localized solutions, we set $f(\xi) = sech(\xi)$, and look for allowed values of A, B, C and D for which (5.13) is solution of equation (5.12). The consistency conditions leads to: $A = 8\theta_4\tilde{A} - 3\theta_3$, $B = 8\theta_4\tilde{B} - 3\theta_3\tilde{D}$, $C = 8\theta_4$ and $D = 8\theta_4\tilde{D}$ where $\tilde{A}$, $\tilde{B}$, $\tilde{D}$ and α are given by: $(1024{\theta_4}^4)\tilde{A}^3 + (1536\theta_2{\theta_4}^3 + 432{\theta_3}^3\theta_4{}^2 - 864{\theta_3}^2{\theta_4}^2)\tilde{A} + (-3{\theta_3}^4 + 162{\theta_3}^3\theta_4 - 576\theta_2\theta_3{\theta_4}^2 - 768k{\theta_4}^2) = 0$, $\tilde{B} = \frac{-\tilde{D}(-54{\theta_3}^2+27{\theta_3}^3+96\theta_2\theta_4+128{\theta_4}^4\tilde{A}^2)}{64\tilde{A}{\theta_4}^2}$, $\tilde{D} = \pm\frac{8\sqrt{2}\tilde{A}\theta_4}{\sqrt{54{\theta_3}^2-27{\theta_3}^3-96\theta_2\theta_4-64{\theta_4}^2\tilde{A}^2}}$ and $\alpha^2 = \frac{18{\theta_3}^2-9{\theta_3}^2-32\theta_2\theta_4-64\tilde{A}{\theta_4}^2}{8{a_1}^2\theta_4}$.

Since the exact closed form solution is known, a simple maxima-minima analysis is sufficient to distinguish parameter regimes supporting dark/bright solitary waves. In this case, when $AD > BC$ one gets a bright soliton, whereas if $AD < BC$ then dark soliton exists. Figure (5.1) depicts intensity profile of a typical dark soliton. It is interesting to note that for dark solitons, from equation (5.11), the kinematic chirping is dominant at the centre of the pulse whereas the higher order chirping is

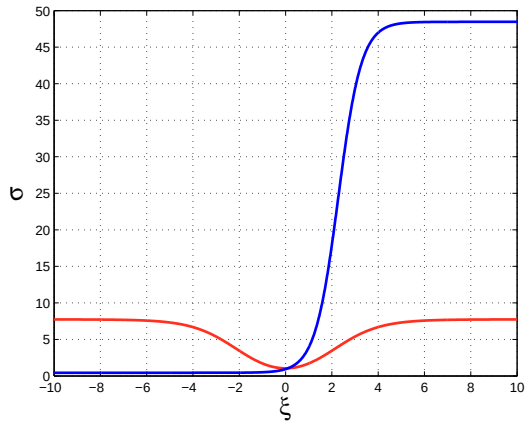

Figure 5.1: *Intensity profile of few solutions: i) Dark Soliton (in red color) for* $a_1 = 1.6001$, $a_2 = -2.6885$, $a_3 = -2.8302$, $a_4 = 0.30814$, $a_5 = 0.76604$, $u = 4.1185$, $c = -3.1186$, $k = -77.965$, $A = 35.36$, $\alpha = 4.7421$, $B = -11.912$, $C = 4.5702$ *and* $D = 17.855$*; ii) Kink Soliton (in blue color)* $a_1 = -11.197$, $a_2 = 44.778$, $a_3 = 6.219$, $a_4 = 19.066$, $a_5 = 37.301$, $u = 884.36$, $c = -13810$, $k = 4360.2$, $A = 0.94014$, $\alpha = 8.6548$, $B = 0.06745$ *and* $D = -0.97921$.

dominant away from the center (see figure (5.2)). However, exactly the opposite is true for bright solitons where the center is dominated by higher order chirping and kinematic chirping is important away from the center. This shows that there is duality between dark soliton-kinematic chirping and bright soliton-higher order chirping. A mutual cancellation will occur at some point(s) when both kinematic chirping and higher order chirping are comparable and have opposite signs, and will result in chirp reversal at the point(s) of cancellation.

Chirp reversal plays a significant role in fiber optics, and has attracted considerable attention recently in context of pulse retrieval in dispersion-nonlinearity managed optical fibres [148, 149, 150, 151]. Chirp reversal occurs at $\xi_r = \pm\cosh^{-1}(\frac{D\sigma_c - B}{A - \sigma_c C})$ provided $\frac{-4c}{3a_4+2a_5} > 0$ and $\frac{D\sigma_c - B}{A-\sigma_c C} \geq 1$, where $\sigma_c = \sqrt{\frac{-4c}{3a_4+2a_5}}$. We have plotted χ' against ξ in figure (5.2) where the chirp reversal is clearly seen as two maxima in the profile.

It should be noted that equation (5.12) with $\theta_4 \neq 0$ has no kink solutions, which are of the type:

$$\sigma = \frac{A + B\tanh(\xi)}{C + D\tanh(\xi)}. \tag{5.14}$$

However, for $a_4 = 1/4$, θ_4 is zero allowing existence of these kind of solutions. In this case, $k = -\frac{({\theta_1}^2+2{\theta_2}^2)}{3\theta_3}$, $A = -\frac{-2(\theta_1+\theta_2)\pm\sqrt{12({\theta_1}^2-{\theta_2}^2)-18\theta_2 k}}{3\theta_3}$, $B = \Gamma D$ where $\Gamma = -\frac{(2\theta_1 A+3\theta_3 A^2+8\theta_2 A-6k)}{6\theta_3 A+4\theta_2-2\theta_1}$, $C = 1$

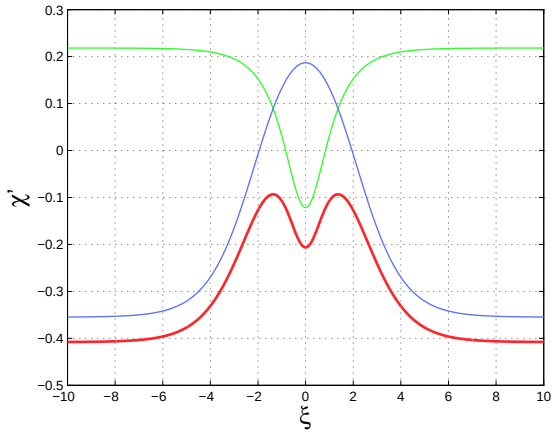

Figure 5.2: *Phase profile of dark soliton plotted in Figure 1. (in red color). Blue curve shows the contribution from linear chirp whereas green curve shows the contribution from higher order chirp. The chirp reversal is clearly seen as peaks in the red curve.*

and $D = \pm\sqrt{\frac{2k-4\theta_2 A-3\theta_3 A^2}{2\theta_1(A-\Gamma)}}$. Figure (5.1) shows intensity profile of a typical kink solution. These solutions being asymmetric around $\xi = 0$, have interesting phase profile which show chirp reversal only once along the profile. The condition for existence of this reversal is given by: $\frac{-4c}{3a_4+2a_5} > 0$ and $-1 \leq \frac{A-\sigma_c C}{D\sigma_c-B} \leq 1$, and the point of reversal is $\xi_r = \tanh^{-1}(\frac{A-\sigma_c C}{D\sigma_c-B})$.

Solutions to equation (5.12) via (5.13) are not restricted to localized ones alone, periodic solutions also exist. Infact it is easy to show that:

$$\sigma = \frac{-3\theta_3}{8\theta_4} \pm \frac{C\cos(\xi)}{\sqrt{2} \pm \cos(\xi)} \tag{5.15}$$

is a periodic solution of equation (5.12), provided ${\theta_1}^3+2{\theta_1}^2(2\theta_2-\frac{9{\theta_3}^2}{8\theta_4}+\frac{9{\theta_3}^2}{16\theta_4})+\frac{8\theta_4}{3}(\frac{-3\theta_2\theta_3}{2\theta_4}+\frac{27{\theta_3}^3}{64\theta^2}-\frac{{\theta_3}^4}{128{\theta_4}^3})^2 = 0$ and $C = -2{\theta_1}^2[2\theta_2 - \frac{9{\theta_3}^2}{8\theta_4} + \frac{9{\theta_3}^2}{16\theta_4}]$.

Apart from the solutions discussed above, amplitude equation (5.12) albeit with different parameters, has been carefully studied in context of cubic quintic nonlinear Schrödinger equation [152, 153]. It has been shown that this equation possesses a rich solution space, where the solutions are expressible in terms of Weierstrass functions, and nature of the solution crucially depends upon initial conditions. Similar analysis for this system would be relevant, and will shed light on the structure of solution space.

A natural question arises whether the model is integrable in this regime or not. Following the Ablowitz-Ramani-Segur algorithm, we investigate singularity structure of the ordinary differential

equation (5.12), which is obtained from an exact reduction of the original partial differential equation (5.7) [154, 155]. Interestingly, we found that ordinary differential equation represented by (5.12) possess poles as the only movable singularities, which implies that this system indeed has the P-property [154, 155]. Hence, we see that this system passes Painlevé test, and is Painlevé integrable, which guarantees existence of regular solutions in general.

5.4 Conclusion

In this chapter, we show a phenomenological modelling of pulse propagation in a nonlinear optical fibre. We found a new class of exact solutions to NLSE system in the presence of self steepening and self frequency shift terms. These include localized solutions of dark-bright type, kink solutions and periodic solutions. These solutions have nontrivial phase chirping which varies as a function of intensity and are different from that in Ref. [143] where the solutions had a trivial phase. A nontrivial connection of this sytem with linear Schrödinger equation in appropriate limits is pointed out. A duality is seen between the dark-bright type of solution and kinematic-higher order chirping. A novel form of self phase modulation has been observed in this case, which shows chirp reversal across the pulse profile. It is known that prechirping of pulses often leads to better quality of pulses, in particular it is quite effective with distributed GVD and nonlinearity [148, 156, 157, 151]. In this context, the solutions having chirping due to initial conditions as well as dynamical conditions will provide a better control. It is noted that for some parameter values the intensity and phase of these solitons will exactly be the same as NLSE solitons, and are found to be chiral, with the direction of propagation controlled by self steepening term. Both dark-bright solitons are found to exist in normal-anomalous dispersion regimes. It is seen in some cases that intensity of these solitons would be like NLSE soliton and only phase structure will be different. Kink solutions are found to exist in this system for special choice of parameters. The system is seen to possess P-property and hence is Painlevé integrable.

Appendix

The Lagrangian for the SU(2) Yang-Mills-Chern-Simons theory (YMCS) is given by

$$\mathcal{L}_{YMCS} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{M}{4}\epsilon^{\mu\nu\rho}[F^a_{\mu\nu}A^a_\rho - \frac{g}{3}\epsilon^{abc}A^a_\mu A^b_\nu A^c_\rho], \tag{5.16}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu$ is the field strength tensor, M the topological mass of the gauge field A, and g is the coupling constant. The ϵ's are the completely antisymmetric structure constants of the gauge group.

As it stands Lagrangian (5.16) is not explicitly gauge invariant, since it changes by a total derivative under a gauge transformation

$$S_{YMCS} = \int d^3x\, \mathcal{L}_{YMCS} \longrightarrow \int d^3x\, \mathcal{L}_{YMCS} + \frac{8\pi^2 M}{g^2}W(U), \tag{5.17}$$

where $W(U)$ is the winding number:

$$W(U) = \frac{1}{24\pi^2}\int d^3x\epsilon^{\alpha\beta\gamma}\mathrm{Tr}\left[\partial_\alpha UU^{-1}\partial_\beta UU^{-1}\partial_\gamma UU^{-1}\right].$$

Here $U(x) = \exp[i\sigma^a\lambda^a(x)]$ is the non-Abelian gauge transformation and σ^a are the Pauli matrices. Demanding the gauge invariance of the partition function, results in the quantization of the CS parameter M:

$$M = \frac{g^2 n}{4\pi}, \quad n \in Z. \tag{5.18}$$

The YMCS gauge field can be coupled to an external source field by augmenting Lagrangian $\mathcal{L}_{YMCS}$ with an interaction term:

$$\mathcal{L}_S = A^a_\mu J^{\mu\, a},$$

to give the classical Euler-Lagrange equation:

$$D_\nu F^{\nu\mu a} + \frac{M}{2}\epsilon^{\mu\nu\rho}F^a_{\nu\rho} = J^{\mu\, a}; \tag{5.19}$$

where $D_\mu F^a_{\nu\rho} = \partial_\mu F^a_{\nu\rho} + g\epsilon^{abc}A^b_\mu F^c_{\nu\rho}$.

It needs to be pointed out, that the problem of solving (5.19) is analogous to classical electrodynamics were one solves Maxwell's equation of motion for a given current density distribution, to yield electric and magnetic fields as a function of space-time. However, crucial difference lies in the fact, that above theory being non-Abelian in nature, has self-coupling in built. This is manifested

in (5.19) as occurrence of nonlinear self interacting term. Such a term is absent in Maxwell's theory, as result Maxwell's equation of motion are linear and hence easily soluble say using Greens function technique for any arbitrary charge distribution. Further, Maxwell equations (in vacuum) being linear equations can not offer special solutions like localised solitons or chaos, which can occur in equations of above kind atleast in principle. Equations (5.19) are coupled nonlinear PDEs and have infinite degrees of freedom. Ideally, it is desirable to have exact solutions to these equations, so that a complete understanding of the classical phase space is obtained. However, there are only a handful of such theories, which are said to be *integrable*, for which the classical phase space is completely understood, Sine-Gordon model being one of the celebrated ones[135]. In such an integrable theory, there exists an infinite dimensional symmetry algebra, which provides infinite conserved quantities, which leads to exact solubility of the equations of motion.

It is known that symmetry algebra of Yang-Mills-Chern-Simons theory is only finite dimensional, and hence does not belong to the integrable class, like most non-trivial realistic theories. Hence, equation (5.19) in its generality is not exactly soluble. However, the nonintegrable set of equations open up an interesting possibility of having chaos, wherein the evolution of the system shows sensitive dependence on initial conditions, and evolution from any two infinitesimally close points in the phase space leads to exponential separation in the trajectories[136]. It is obvious that, such a behaviour can not be described analytically and numerical evolution methods are used to study such systems. Numerical analysis of equations (5.19), which are coupled nonlinear PDEs, is quite involved and computationally expensive. An alternative way to infer about the dynamics, is by choosing suitable variables, over which one imposes additional constraints which are physically meaningful. This reduces the degrees of freedom of the system, and may render this constrained system solubility. Although such a study would be restrictive, but undoubtedly would throw light on the dynamics of the parent model.

The classical solutions of YMCS field theories with and without a source term has been investigated over the years by a number of authors [158, 159, 160, 161, 162, 163, 164]. Restricting ourselves to purely time dependent situations of (5.19): $A^a_\mu = A^a_\mu(t)$ and $J^{\mu\,a} = J^{\mu\,a}(t)$, and choose ansatz for the fields, along with external source as:

$$A^a_\mu = \delta^a_\mu f_a(t), \quad A^{\mu a} = g^{\mu a} f_a(t), \quad J^{\mu a} = g^{\mu a} j_a(t). \tag{5.20}$$

Here $a = 1, 2, 3$, $\mu = 1, 2, 3$ ($x_1 = t, x_2 = x, x_3 = y$) and $g_{\mu\nu} = \mathrm{diag}(+, -, -)$. Note that there is

no summation for a. With above choice of ansatz, we naturally fix our gauge choice to Coulomb gauge condition whereby: $\vec{\nabla} \cdot \vec{A} = 0$.

With above mentioned ansatz, the equations of motion can be written in explicit component form as:

$$\begin{aligned} Mgf_2f_3 + g^2 f_1(f_2^2 + f_3^2) &= j_1, \\ Mgf_1f_3 - \ddot{f_2} + g^2 f_2(f_1^2 - f_3^2) &= j_2, \\ Mgf_1f_2 - \ddot{f_3} + g^2 f_3(f_1^2 - f_2^2) &= j_3, \\ M\dot{f_2} + g(\dot{f_1}f_3 + 2f_1\dot{f_3}) &= -j_2, \\ M\dot{f_3} + g(\dot{f_1}f_2 + 2f_1\dot{f_2}) &= -j_3, \end{aligned} \tag{5.21}$$

where $\dot{f_a}$ denotes derivative with respect to time. Notice, that this system has only finite number of degrees of freedom, and hence finite dimensional phase space. Solutions to above system of coupled ordinary differential equations, would yield restrictive solutions to equation (5.19). To this end let us set $f_1 = \kappa$ (a constant) and take $f_2 = f_3$. With $j_1 = 0$ the first equation in the above set gives $\kappa = -M/2g$. Setting $j_2 = j = j_3 = j$, we get

$$\ddot{f}(t) + \frac{M^2}{4} f(t) + g^2 f(t)^3 + j(t) = 0, \tag{5.22}$$

which when solved would give information about dynamics of this model. In static case, i.e., when both f and j are constant, is easy to see that $f = 0$ is not a solution for non-zero current. This is particularly interesting since it means, subject to assumed constraints, in presence of constant background current density gauge field can not be zero. When $j(t)$ is absent this equation reduces to an elliptic differential equation, whose solutions can be expressed in terms of elliptic functions. Unlike linear differential equations, the above equation does not allow general solution for any arbitrary function j. Instead, only special solutions for certain form of j can be found. Considering j to be a constant, it has been shown that the exact solutions to the above system can be found[165, 166]. Further, in this case, solution to above equation is related to Jacobi elliptic functions via a conformal (Möbius) transformation, which amounts to say that:

$$f(t) = \frac{A + B\ \phi(t,m)}{C + D\ \phi(t,m)}. \tag{5.23}$$

Here A, B, C, and D are real constants, such that $AD - BC \neq 0$, and $\phi(t,m)$ is one of the twelve Jacobi elliptic functions, which satisfy a differential equation of the kind:

$$\ddot{\phi} + a\phi + b\phi^3 = 0, \tag{5.24}$$

with a and b being real constants. These Jacobi elliptic functions are generalization of circular and hyperbolic trigonometric functions[154]. Each of these functions, are characterised by a modulus parameter m, for example, the elliptic function $cn(x,m)$ goes to $cos(x)$ in the limit $m \to 0$, and to $sech(x)$ when $m \to 1$. Although for $m > 1$ these elliptic functions are defined, they can be rewritten in terms of elliptic functions with $0 < m < 1$. Defining,

$$E_0 = \frac{\dot{\phi}^2}{2} - \frac{a\phi^2}{2} - \frac{b\phi^4}{4}, \tag{5.25}$$

and substituting (5.23) in equation (5.22), one ends up to get consistency conditions in equation parameters and the solution parameters:

$$-4BCDE_0 + 4AD^2E_0 + \frac{M^2}{4}AC^2 + g^2A^3 + jC^3 = 0, \tag{5.26}$$

$$aBC^2 - aACD + \frac{M^2}{4}BC^2 + \frac{M^2}{2}ACD + 3g^2A^2B + 3jC^2D = 0, \tag{5.27}$$

$$-aBCD + aAD^2 + \frac{M^2}{2}BCD + \frac{M^2}{4}AD^2 + 3g^2AB^2 + 3jCD^2 = 0, \tag{5.28}$$

$$bBC^2 - bACD + \frac{M^2}{4}BD^2 + g^2B^3 + jD^3 = 0 \tag{5.29}$$

These equations being linear in equation parameters can be solved in terms of solution parameters to give:

$$E_0 = -\frac{bAC^3 + aBC^2D + aACD^2}{4BD^3}, \tag{5.30}$$

$$M^2 = 4\frac{-3bAC^2 - 2aBCD - aAD^2}{D(-BC + AD)}, \tag{5.31}$$

$$g^2 = \frac{-bC^3 - aCD^2}{B(BC - AD)}, \tag{5.32}$$

$$j = -\frac{bABC^2 + bA^2CD + aB^2CD + aABD^2}{D^2(BC - AD)}. \tag{5.33}$$

By choosing various values for A, B, C & D, we get a spectrum of values for the equation parameters for which above considered ansatz is a solution. For example, for $\phi = cn(t,m)$, $a = 2m - 1$ and $b = -2m$, and above relations with these values determine the equation parameters for which this ansatz is a solution. Interestingly, for $m = 1$, this choice of ansatz yields localized solitons, which are of the type:

$$f = \frac{A + B\ Sech(t)}{C + D\ Sech(t)}. \tag{5.34}$$

When $AD > BC$ one gets dark solitons (wherein the function has a minimum below the constant background), and when $AD < BC$ one gets bright solitons(wherein function has a maximum above

a constant background). On the other hand, when $m = 0$, one gets periodic solutions of the kind:

$$f = \frac{A + B\ Cos(t)}{C + D\ Cos(t)}. \tag{5.35}$$

Hence, various possible choice of elliptic functions and possible values for A, B, C & D, which ensure the positivity of M^2 and g^2, yield various valid solutions to equation (5.22). An interesting point to notice is that, the solutions are necessarily defined over a non-zero constant background $\frac{A}{C}$, which is present as a consequence of a constant source term. Hence, the solutions found above do not obey equation (5.19) with the usually assumed boundary conditions $A^{\mu a} \to 0$ as $x, y \to \pm\infty$.

It is interesting to ask, whether this model (5.22) exhibits chaos or not, for some nontrivial $j(t)$. For, $j(t) = Cos(t)$, equation (5.22) resembles the famous forced Duffing oscillator, which is a standard example to demonstrate chaotic dynamics. Positivity of M^2 and g^2, does not allow this model to exhibit chaos, which requires any one of these to be negative. Hence, we conclude that this model, defined by (5.22), can not be made chaotic by coupling it with an external source.

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